Algebra 612 **Prelim Exercises on Fields and Galois Theory**

1. Find the Galois group of $x^4 - 2$ over (a) $\mathbb{Q}$ (b) $\mathbb{F}_3$ (c) $\mathbb{F}_7$.

   Find the Galois group of $x^4 + 2$ over (a) $\mathbb{Q}$ (b) $\mathbb{F}_3$ (c) $\mathbb{F}_5$.

2. Let E/F be a finite Galois extension. Let $G = \mathrm{Gal}(E/F)$, $K = \{u \in E \mid \sigma\tau(u) = \tau\sigma(u)\ \forall \sigma, \tau \in G\}$. Prove:

   a) K is an intermediate subfield.

   b) K/F is a Galois extension with abelian Galois group.

3. Let ζ be a primitive 20th root of unity in $\mathbb{C}$, and let $E = \mathbb{Q}(\zeta)$.

   a) Identify the Galois group $\mathrm{Gal}(E/Q)$, explaining how the individual automorphisms act on E.

   b) How many subfields of E are there which are quadratic extensions of $\mathbb{Q}$?

   c) Determine the irreducible polynomial of ζ over $\mathbb{Q}$.

4. Let K be the splitting field of $x^6 - 25$ over $\mathbb{Q}$. Determine $\mathrm{Gal}(K/\mathbb{Q})$. Explicitly determine all subfields of K, giving generators over $\mathbb{Q}$. Indicate which are Galois over $\mathbb{Q}$.

5. Under what conditions on q is the polynomial x^2+x+1 irreducible over a finite field F with q elements? (Hint: consider the multiplicative group of nonzero elements of F.)

6. Let $K = \mathbb{Q}(\alpha)$, where $\alpha^3 = 5$. Determine the mimimal polynomial of $\alpha + \alpha^2$ over $\mathbb{Q}$.

7. Consider $f(x) = x^4 + x^3 + x^2 + x + 1$ as a polynomial over $\mathbb{Q}$. How many subfields are there of the splitting field of f? Justify your answer.

8. Let K be a field containing exactly 64 elements. Find all the subfields of K. For how many elments $\alpha \in K$ is $K = \mathbb{F}_2(\alpha)$? How many irreducible polynomials of degree 6 are there in $\mathbb{Z}_2[x]$?

9. Let $f(x) = x^5 - 5x^2 + 1$. Show f has precisely three real roots and is irreducible over $\mathbb{Q}$. Let $G = \mathrm{Gal}(f)$, the Galois group of f over $\mathbb{Q}$. Show G contains a 5-cycle and a 2-cycle. What is the Galois group of f? Is f solvable by radicals? Explain.

10. Let K be the splitting field of the function $f(x) = x^4 - 5$ over $\mathbb{Q}$. Find the Galois group G of f, and list all the subgroups of G and the corresponding fixed subfields.

11. Let L/K be a Galois extension of fields, with Galois group $G = \{\sigma_1, \dots, \sigma_n\}$, and let $\alpha \in L$. Show that $L = K(\alpha)$ if and only if $\sigma_1(\alpha), \dots, \sigma_n(\alpha)$ are distinct.

12. Let $c \in F$, where F is a field of characteristic $p > 0$. Prove that $x^p - x - c$ is irreducible in $F[x]$ if and only if $x^p - x - c$ has no root in F. Show this is false if F is of characteristic 0.

13. Find the Galois group of $x^4 - 10x^2 + 5$ over the rationals.

14. Let p be a prime. Demonstrate the existence of a Galois extension of $\mathbb{Q}$ whose Galois group is cyclic with p elements.

15. Let K be a field of characteristic 0 in which every cubic polynomial has a root. Let $f(x)$ be an irreducible quartic polynomial with coefficients in K whose discriminant is a square in K. What is the Galois group of $f(x)$ over K?

16. Let $f(x) = x^p - x - c$ where p is a prime not dividing $c \in \mathbb{Z}$. Show that $f(x)$ is irreducible over $\mathbb{Q}$. (Hint: try the following steps: Show that if $f(x)$ is irreducible over $\mathbb{F}_p$, then it is irreducible over $\mathbb{Q}$. Show that $f(x)$ does not have a root in $\mathbb{F}_p$. Then consider problem 12 above.)

17. Let F be a field. SHow that if $[F9\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

18. Prove: A field K is perfect if and only if every finite extension of K is separable. Do not just quote results from class. Give the details of the proof.

19. A field F is called *formally real* if -1 is not expressible as a sum of squares in F. Let $f(x) \in F[x]$ be an irreducible polynomial of odd degree, and let α be a root of $f(x)$. Prove that if F is formally real, then $F[\alpha]$ is also formally real.