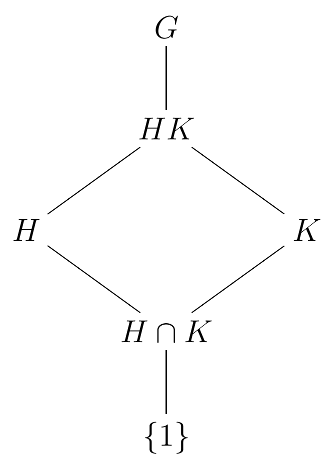
5 The Isomorphism Theorems

The first of the isomorphism theorems is basically an observation we have already made, since for any group homomorphism ϕ, it is easy to see that the map $G/\ker(\phi) \to \phi(G)$ gives a well-defined bijective homomorphism.

Theorem 5.1 (First Isomorphism Theorem) *If $\phi : G \to H$ is a homomorphism of groups, then $\ker(\phi) \trianglelefteq G$ and $G/\ker(\phi) \cong \phi(G)$.*

Theorem 5.2 (Second, or "Diamond", Isomorphism Theorem) *G a group, H, K subgroups of G with $H \leq N_G(K)$. Then $H \cap K \trianglelefteq H$, and $HK/K \cong H/(H \cap K)$.*



Proof. HK is a subgroup of G, since $H \leq N_G(K)$, and this also shows $K \trianglelefteq HK$. Now let $h \in H, x \in H \cap K$. Then $hxh^{-1} \in H$, and since H normalizes K and $x \in K$, also $hxh^{-1} \in K$. Thus $hxh^{-1} \in H \cap K$, and $H \cap K \trianglelefteq H$.

Define $\phi : HK \to H/(H \cap K)$ by $\phi(hk) = h(H \cap K)$. Need to check ϕ is well-defined (if $hk = h_1k_1$, show $h(H \cap K) = h_1(H \cap K)$), ϕ is a homomorphism ($\phi(hk)\phi(h_1k_1) = \phi(hkh_1k_1)$ – to see this, observe $hkh_1k_1 = hh_1(h_1^{-1}k_1h_1)k_1$), ϕ is onto (straightforward). Then observe

$$\begin{aligned} \ker(\phi) &= \{hk \mid \phi(hk) = 1(H \cap K)\} \\ &= \{hk \mid h(H \cap K) = H \cap K\} \\ &= \{hk \mid h \in H \cap K\} = K, \end{aligned}$$

since $1 \in H \cap K$¡ so $K \subseteq \ker(\phi)$, and clearly $\ker(\phi) \subseteq K$. Finally, apply the First Isomorphism Theorem.

Theorem 5.3 (Third Isomorphism Thm, or "Invert & Multiply") *G a group, $H \trianglelefteq G, K \trianglelefteq G, H \leq K$. Then $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$.*

Proof. $K/H \trianglelefteq G/H \iff (gH)(kH)(gH)^{-1} \in K/H \iff (gkg^{-1}H \in K/H \quad \forall gH \in G/H, kH \in K/H$, which is true because $gkg^{-1} \in K$, since $K \trianglelefteq G$. Then define $\phi : G/H \to G/K$ by $gH \mapsto gK$. It is well defined because $H \leq K$, clearly onto, and the kernel is $\{gH \mid gK = K\} = \{gH \mid g \in K\} = K/H$.

Theorem 5.4 Fourth, or "Lattice", Isomorphism Theorem *G a group, $N \trianglelefteq G$. Then there exists a one-one correspondence between subgroups H of G containing N and subgroups $\overline{H} = H/N$ of G/N, so all subgroups of G/N are of the form H/N, where $H \geq N$ is a subgroup of G. Moreover, for all $H, K \leq G$ with $N \leq H, N \leq K$, we have*

1. *$H \leq K \iff \overline{H} \leq \bar{K}$,*
2. *$H \leq K \implies [K : H] = [\overline{K} : \overline{H}]$,*
3. *$\overline{\langle H, K \rangle} = \langle \overline{H}, \overline{K} \rangle$,*
4. *$\overline{H \cap K} = \overline{H} \cap \overline{K}$, and*
5. *$H \trianglelefteq G \iff \overline{H} \trianglelefteq \overline{G}$.*

Proof. First observe that the complete preimage of a subgroup in G/N is a subgroup of G, which obviously contains N. Everything else is a straightforward check.

The Fourth Isomorphism Theorem means that the lattice for G/N appears in the lattice for G as that section of the lattice that "sits above" N. Look at examples with normal subgroups of Q_8, D_8.

Notice also that there are times when one wants to specify a homomorphism ϕ on a quotient group G/N based on a given homomorphism Φ on the group G, by declaring $\phi(gN) = \Phi(g)$, and one asks whether this is "legal". The answer is yes if and only if $N \leq \ker(\Phi)$, in which case we say Φ *factors through* N, and refer to ϕ as the *induced homomorphism* on G/N. Look at commutative diagram associated with this.