7 Composition Series and Solvable Groups

It is possible to have groups G, G' with normal subgroups N, N' such that $N \simeq N', G/N \simeq G'/N'$, but $G \not\simeq G'$. However, it is often the case that knowing something about $N, G/N$ can give information about G. For example, we had Exercise 36 in §3.1: If $G/Z(G)$ is cyclic, then G is abelian. There is a kind of "induction" that is often used in arguments in group theory. If a property P holds for all groups of order less than the order of G, and if whenever there exists a normal subgroup $N \trianglelefteq G$ such that $N, G/N$ satisfying P implies G satisfies P, then if G has a nontrivial proper normal subgroup, G must satisfy P. (As an example, you can read Proposition 21 in §3.4 of the text. It is a special case of Cauchy's Theorem, which you proved as a homework assignment.)

The point is that to understand a finite group, one method is to try to understand its normal subgroups and its quotient groups and build inductively. From that viewpoint, the sticking points are the finite simple groups. Essentially we are trying to study "factorization" of groups.

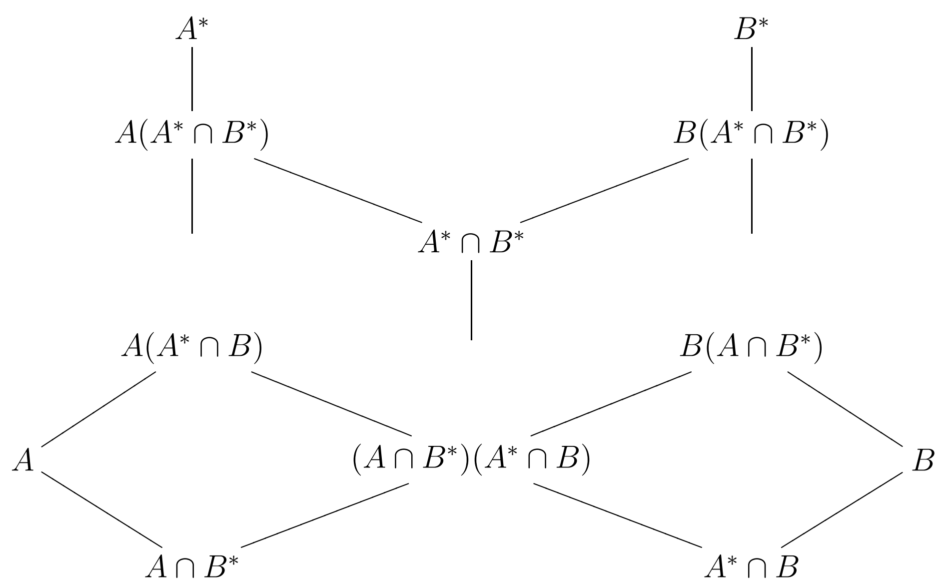
Definition 7.1 *A* subnormal series *of a group G is a chain of subgroups $G = G_n > G_{n-1} > \cdots > G_0$ such that $G_i \triangleleft G_{i+1}, 0 \leq i \leq n-1$. (Some authors call this a em normal series.) The* factors *of the series are the quotient groups G_{i+1}/G_i, and the* length *of the series is the number of strict inclusions (nonidentity factors). A subnormal series such that $G_i \triangleleft G$ for all i is said to be* normal. *A subnormal series such that $G_0 = \{1\}$ and each factor is simple is a* composition series. *The simple quotient groups are the* composition factors *of G. Given a subnormal series $1 = N_0 \leq \cdots \leq N_r = G$, the subnormal series $1 = M_0 \leq \cdots \leq M_s$ is a refinement of it if the list $N_0, \ldots, N_r$ is a sublist of $M_0, \ldots, M_s$. Two subnormal series for G are* equivalent *if there is a one-to-one correspondence between their nontrivial factor groups such that corresponding factor groups are isomorphic.*

Theorem 7.2 (Jordan-Hölder) *Let G be a nontrivial finite group. Then G has a composition series, and any two composition series for G are equivalent.*

Proof. To see that every finite group has a composition series, let G_1 be a maximal normal subgroup of G. Then G/G_1 is simple. Let G_2 be a maximal normal subgroup of G_1, etc. Since in each case $1 \leq |G_i| \leq |G_{i-1}|$, the process must stop: $G = G_0 \geq G_1 \geq \cdots \geq G_n = \{1\}$. To see that any two composition series are equivalent, we need the following technical lemma:

Theorem 7.3 (Zassenhaus, 1935): **Butterfly Lemma**) *Let* $A \lhd A^*, B \lhd B^*$ *be subgroups of a group* G*. Then* $A(A^* \cap B) \lhd A(A^* \cap B^*), B(B^* \cap A) \lhd B(B^* \cap A^*)$*, and*

$$\frac{A(A^* \cap B^*)}{A(A^* \cap B)} \cong \frac{B(B^* \cap A^*)}{B(B^* \cap A)} \cong \frac{A^* \cap B^*}{(A \cap B^*)(A^* \cap B)}.$$



Proof. Let $D = (A^* \cap B)(A \cap B^*)$. Now $B \lhd B^* \implies B(B^* \cap A^*) \leq B^*$ and $B \lhd B(B^* \cap A^*)$. The second isomorphism theorem (with K corresponding to B, H corresponding to $B^* \cap A^*$) shows $(B \cap A^*) \lhd (B^* \cap A^*)$. Similarly, $(A \cap B^*) \lhd (B^* \cap A^*)$. Then $D \lhd (A^* \cap B^*)$.

If $x \in B(B^* \cap A^*)$, write $x = bc, b \in B, c \in B^* \cap A^*$. Define $f : B(B^* \cap A^*) \to (A^* \cap B^*)/D$ by $f(x) = f(bc) = cD$. We need to check that f is well defined. If $x = bc = b_1c_1$, then $c_1c^{-1} = b_1^{-1}b \in B \cap (B^* \cap A^*) = B \cap A^* \subseteq D$. Then check that f is a homomorphism, is onto, and that the kernel is $B(B^* \cap A)$. Then the first isomorphism theorem says

$$\frac{B(B^* \cap A^*)}{B(B^* \cap A)} \cong \frac{A^* \cap B^*}{D}.$$

The whole set-up is symmetric in A and B, which also gives

$$\frac{A(A^* \cap B^*)}{A(A^* \cap B)} \cong \frac{A^* \cap B^*}{D}$$

.

Theorem 7.4 (Schreier) *Any two subnormal series of an arbitrary group G have equivalent refinements.*

Proof. Let $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1\}$ and $G = H_0 \supset H_1 \supset \cdots \supset H_m = \{1\}$ be two subnormal series for G. In the first series, between each G_i and G_{i+1}, insert a "copy" of the second series: $G_{ij} = G_{i+1}(G_i \cap H_j), 0 \leq j \leq m$, so

$$G_{ij} = G_{i+1}(G_i \cap H_j) \supset G_{i+1}(G_i \cap H_{j+1}) = G_{i,j+1}.$$

Since $H_0 = G$, we have $G_{i0} = G_i$, and since $H_m = \{1\}$, we have $G_{im} = G_{i+1}$. Let $A = G_{i+1}, A^* = G_i, B = H_{j+1}, B^* = H_j$ in the Butterfly Lemma. hen we see $G_{i,j+1} \lhd G_{ij}$. Thus the groups G_ij give a subnormal series which is a refinement of the first series above, having $(m+1)(n+1)$ terms. Now do the same for the second series, i.e. set $H_{ij} = H_{j+1}(H_j \cap G_i), 0 \leq i \leq n$. Then $H_{i+1,j} \lhd H_{ij}$, and we get a refinement of the second series above, having $(m+1)(n+1)$ terms. Finally, letting $G_{ij}/G_{i,j+1}$ correspond to $H_{ij}/H_{i+1,j}$, we see we have a one-one correspondence of composition factors, where corresponding factors are isomorphic. Indeed, from the Butterfly Lemma we have

$$\frac{G_{ij}}{G_{i,j+1}} = \frac{G_{i+1}(G_i \cap H_j)}{G_{i,j+1}} \cong \frac{H_{j+1}(G_i \cap H_j)}{H_{j+1}(G_{i+1} \cap H_j)} = \frac{H_{ij}}{H_{i+1,j}}.$$

The existence of composition series and the fact that the composition factors are unique is kind of like a "unique factorization" property for groups. To understand all finite groups, we need to understand the finite simple groups (like understanding the prime factors), and then understand how smaller groups can be put together to form new groups, i.e., solve the "extension" problem: given groups A and B, what groups C can fill out the short exact sequence

$$1 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 1.$$

Observe that the only simple abelian groups are the groups $\boldsymbol{Z}/p\boldsymbol{Z}$, where p is a prime. (More amazing is the celebrated result of Feit-Thompson, which says that these are also the only possible simple groups of odd order!) Groups whose composition factors are all abelian turn out to be a particularly important class of groups.

Definition 7.5 *A group G is* solvable *if it has a subnormal series $G = G_0 \geq \cdots \geq G_n = \{1\}$ such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Such a series is called a* solvable series *for G.*

Theorem 7.6 *Let G be a finite group. Then G is solvable if and only if all of the composition factors for G have prime order.*

Proof. The reverse implication is clear. Conversely, given a solvable series for G, if $G_{i+1} \neq G_i$, let H_{i1} be a maximal normal subgroup of G_i containing G_{i+1}. If $H_{i1} \neq G_{i+1}$, let H_{i2} be a maximal normal subgroup of H_{i1} containing G_{i+1}. Continue. Since G is finite, this process must terminate for each i: $G_i > H_{i1} > H_{i2} > \cdots > H_{ik_i} > G_{i+1}$, with each subgroup a maximal normal subgroup of the preceding one, there for simple. $H_{ij}/H_{i,j+1} \cong (H_{ij}/G_{i+1})/(H_{i,j+1}/G_{i+1})$, and each factor here is a subgroup of G_i/G_{i+1}, hence abelian, so $H_{ij}/H_{i,j+1}$ is abelian and simple, hence of prime order.

Theorem 7.7 *Let G be a group. The following are equivalent:*

1. *G is solvable.*
2. *G has a normal subgroup N such that N and G/N are solvable.*
3. *Every subgroup and every quotient group of G is solvable.*

Proof. It is clear that (3) implies (1) and (1) implies (2). We will show that (2) implies (1) and (1) implies (3) for finite groups – the more general statement is true, but would take us too far afield to prove right now.

Let N be a normal subgroup of G such that N and G/N are both solvable. Let $G/N = K_0^* > K_1^* > \cdots > K_m^* = \{1\}$ be a solvable series for G/N, and let $N = N_0 > N_1 > \cdots > N_r$ be a solvable series for N. Then by the correspondence of subgroups K_i^* of G/N with subgroups K_i of G containing N, we have $G = K_0 > K_1 > \cdots > K_m = N > N_1 > \cdots > N_r = \{1\}$ is a subnormal series for G, with $K_i/K_{i+1} \cong K_i^*/K_{i+1}^*$ abelian, giving a solvable series for G.

If G is a finite solvable group, the G has a composition series with prime order factors. Let N be a normal subgroup of G, and consider the (sub)normal series $G > N > \{1\}$. By Schreier's Theorem, this series has a refinement which is equivalent to any composition series. In particular, there exists a solvable series for G which has N as one of its terms. Then the part "below" N gives a solvable series for N, and the part "above" N provides a solvable series for G/N when quotients are taken. This shows the truth of statement (3) for finite groups and for normal subgroups.

To see (3) for subgroups in general, observer that if $H < G$, then given a solvable series $G = G_0 > G_1 > \cdots > G_n = \{1\}$ for G, the series $H = H_0 \geq (H \cap G_1) \geq \cdots \geq (H \cap G_n) = \{1\}$ is a solvable series for H. (Observe $H \cap G_{i+1} = (H \cap G_i) \cap G_{i+1} \lhd H \cap G_i$, so this is a subnormal series. Also $(H \cap G_i)/(H \cap G_{i+1} \cong G_{i+1}(H \cap G_i)/G_{i+1} \leq G_i/G_{i+1}$, so it is abelian.)