RESISTANCE DIMENSIONS OF BRANCHING PROCESSES IN VARYING ENVIRONMENTS TREES

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Abstract. In this paper we pursue our study, in [5] and [6], on dimensions of some random trees. These dimensions are, for polynomially growing graphs, closely related to the type (transience-recurrence) problem of the simple random walk on such graphs. We give an interval estimation for the mean resistance dimension of branching processes in varying environments trees.

1. Introduction

The resistance dimension d_Ω of a graph is closely related to the type (transience-recurrence) of the random walk on that graph via the crucial criterion of Nash-Williams [8] that relates the type of random walk on an infinite graph G to the effective resistance of the corresponding electric network when a resistance is assigned to every edge of G. See [1] for a comprehensive treatment. The fractal dimension d_f, the resistance dimension d_Ω, and the random walk dimension d_R of polynomially growing graphs were first studied by Telcs in [9]. He proved, under smoothness conditions, that $d_R = d_f + 2 - d_\Omega$. Afterwards Zhou [10] obtained, subject to little relaxed conditions, the same result. In [5] and [6], we considered the fractal and resistance dimensions of some random trees. Consider a graph $G \equiv (V_G, E_G)$, where V_G is the set of vertices (nodes) and E_G is the set of edges. We consider only connected graphs, for which every pair of vertices is connected by a path of edges. We say that two vertices x, y are adjacent, $x \hookrightarrow y$, if they are connected by one edge. A vertex x and an edge e are said to be incident if x is an end-point of e. The degree d_x of a vertex x is the number of edges incident with x. A graph G is said to be locally bounded if $d_x \leq D < \infty$. The graphical distance $d(x,y)$ between two vertices x and y is the number of edges of the shortest path connecting them. Define the ball $B_{x,n}$ to be the set of vertices of distance at most n from the vertex x, and the sphere $S_{x,n}$ to be the set of vertices at distance n from x. Let $b_{x,n}$ and $Z_{x,n}$ be the cardinality of $B_{x,n}$ and $S_{x,n}$, respectively. For convenience, we omit the the reference vertex x when no confusion may exist. Let us use the notation $x_n \sim y_n$ if $\lim_n \frac{x_n}{y_n} = 1$ and $x_n \asymp y_n$

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if $\lim_n \frac{\log x_n}{\log y_n} = 1$. A graph is said to be polynomially growing if there exists $d \geq 1$ such that the cardinality of the ball of radius n is $b_n \asymp n^d$. Consider a finite graph $G \equiv (V_G, E_G)$. The electric network corresponding to G is obtained by assigning a resistance R_{xy} (a conductance $C_{xy} = \frac{1}{R_{xy}}$) to every edge $(x, y) \in E_G$. A random walk on G, starting at a vertex a and ending at a vertex b, is a Markov chain with the set V_G as the state space and transition probabilities, P_{xy}, defined as:

$$P_{xy} = \begin{cases} \frac{C_{xy}}{C_x}, & \text{if } x \hookrightarrow y; \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

where $C_x = \sum_{z \hookrightarrow x} C_{xz}$. If the same resistance is assigned to each edge $(x, y) \in E_G$, a simple random walk (SRW) is obtained and for which

$$P_{xy} = \begin{cases} \frac{1}{d_x}, & \text{if } x \hookrightarrow y; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

If a current $\imath_a$ is injected into a and withdrawn from b, then a potential difference υ_a between a and b exists. The effective resistance, R_{eff}, between a and b is defined as:

$$R_{eff} = \frac{\upsilon_a}{\imath_a} \tag{1.3}$$

Let υ_x denote the voltage at a node x. An equipotential surface, $\mathcal{E}_k$, is defined to be the set of all vertices of equal voltage k, i.e.,

$$\mathcal{E}_k = \{y \in V_G : \upsilon_y = k\}$$

A way to modify an electric network is **shorting**. Shorting a set of vertices into a single vertex is to connect them together with perfectly conducting wires.

Shorting Law. [1, p.100]. Shorting certain sets of nodes together can only decrease the effective resistance of the network between two given nodes.

Obviously, shorting gives a lower bound for the effective resistance of the underlying graph. Shorting the nodes of an equipotential surface doesnt affect the effective resistance between any given two nodes.

If a current $\imath_a$ flows between a and b, an energy $\imath_a^2 R_{eff}$ is supplied to the network, see [1]. A current $\imath_{xy}$, that flows through (x, y), results in a dissipation of energy, which is equal to $\imath_{xy}^2 R_{xy}$. The total energy dissipation between a and b due to the current flow $\imath$ in the network is equal to $\frac{1}{2} \sum_{x,y \in V_G} \imath_{xy}^2 R_{xy}$.

The principle of conservation of energy asserts that the energy supplied is equal to the energy dissipated. In other words,

$$\imath_a^2 R_{eff} = \frac{1}{2} \sum_{x,y \in V_G} \imath_{xy}^2 R_{xy} \tag{1.4}$$

For a unit current flow

$$R_{eff} = \frac{1}{2} \sum_{x,y \in V_G} \imath_{xy}^2 R_{xy} \tag{1.5}$$

Now, consider an infinite electric network associated with an infinite graph $G \equiv (V_G, E_G)$. Let $G^{(n)} \equiv (B_{a,n}, E_{a,n})$ be a sequence of finite subgraphs of G, where $B_{a,n} \subset V_G$ is the set of vertices of distances at most n from a and $E_{a,n} \subset E_G$ is the set of edges such that if $x, y \in B_{a,n}$ and $(x, y) \in E_G$, then $(x, y) \in E_{a,n}$.

Consider, for $n \geq 1$, a random walk on $G^{(n)}$ that starts at a until it reaches $S_{a,n}$, the set of vertices having distance n from a. We inject a current $\imath_a^{(n)}$ into a and withdraw it from $S_{a,n}$ (being shorted to a single node). Define R_n to be the effective resistance of the finite electric network corresponding the finite graph $G^{(n)}$ between a and $S_{a,n}$. The effective resistance $R_{eff} = R_{eff}(G)$ of G is defined to be the limit of the sequence $R_n = R_{eff}(G^{(n)})$, i.e.,

$$R_{eff} = \lim_{n\to\infty} R_n.$$

This limit is well defined, possibly infinite, because R_n is nondecreasing in n. Random walk on an infinite graph is transient if and only if the effective resistance of its corresponding electric network is finite, see [1] or [8].

We turn our attention now to infinite leafless (for every x, $d_x \geq 2$) locally bounded trees. If we assign a unit resistance to every edge of T and R_n denotes the effective resistance of the portion of the tree from the root up to height n, it follows from shorting law that,

$$R_n \geq \sum_{k=1}^{n} \frac{1}{Z_k} \tag{1.6}$$

where Z_n is the number of vertices at distance n from the root. Therefore the divergence of $\sum_{k=1}^{\infty} \frac{1}{Z_k}$ implies the recurrence of SRW on T. Shorting vertices of equal potential does not affect the effective resistance and as such if T is a spherically symmetric tree SST (all the vertices of the same distance from the root have the same degree), then S_k for every k is an equipotential surface and therefore,

$$R_n = \sum_{k=1}^{n} \frac{1}{Z_k} \tag{1.7}$$

Hence SRW on SST is recurrent if and only if $\sum_{k=1}^{\infty} \frac{1}{Z_k}$ is divergent.

If SRW on a tree T is transient with an effective resistance $R_{eff} = R < \infty$, then the effective resistance from S_n to infinity satisfies that $R - R_n \geq \sum_{k=n+1}^{\infty} \frac{1}{Z_k}$. Also, if T is SST then $R - R_n = \sum_{k=n+1}^{\infty} \frac{1}{Z_k}$.

Definition 1 (Resistance dimension d_Ω)**.** The resistance dimension of a graph $G \equiv (V_G, E_G)$ is defined to be:

$$d_\Omega = \begin{cases} 2 - \limsup_{n\to\infty} \frac{\log R_n}{\log n}, & \text{if } R_{eff} = \infty; \\ \\ 2 - \limsup_{n\to\infty} \frac{\log(R-R_n)}{\log n}, & \text{if } R_{eff} = R < \infty. \end{cases}$$

Loosely speaking,

$$R_n \asymp \begin{cases} n^{2-d_\Omega}, & \text{if } R_{eff} = \infty; \\ \\ R - n^{2-d_\Omega}, & \text{if } R_{eff} = R < \infty. \end{cases}$$

It is known that the SRW on the homogeneous tree of degree 2 is recurrent and d_Ω for such tree is

$$d_\Omega = 2 - \limsup_{n\to\infty} \frac{\log \frac{n}{2}}{\log n} = 2$$

while, the SRW on the complete binary tree (every vertex has degree 3 except the root has degree 2) is transient and

$$d_\Omega = 2 - \limsup_{n\to\infty} \frac{\log(\frac{1}{2})^n}{\log n} = \infty$$

In order to identify the phase transition between finite and infinite resistance dimensions we chose to consider random trees of mixed degrees 2 or 3. It turned out that d_Ω for Galton -Watson trees is infinity, while the critical rate of growth is identified for SST's. See theorem 3 below or [5]. Let d_n denote the degree of every vertex of the n^{th} level. The degrees d_n's are assumed to be independent random variables. let $d_n^+ = d_n - 1$. This sequence gives rise to a spherically symmetric random tree, SSRT. We consider now a (**SSRT**) whose d_n^+'s are distributed as: $d_0^+ = d_0 = 1$ a.s. and for $n \geq 1$,

$$d_n^+ = \begin{cases} 1, & \text{with probability } 1 - q_n; \\ \\ 2, & \text{with probability } q_n \end{cases} \tag{1.8}$$

where $0 < q_n < 1$. Let Z_n denote the size of the n^{th} level S_n. Then $Z_1 = 1$ a.s. and for $n \geq 2$, $Z_n = \prod_{k=1}^{n-1} d_k^+$. Hence, $E(Z_n) = \prod_{k=1}^{n-1}(1 + q_k)$.

1.1. **Branching processes in varying environments trees (BPVET).**

Consider a doubly indexed family $\{d_{n,k}^+; n = 0, 1, \ldots; k = 1, 2, \ldots\}$ of random variables. Let $d_{n,k}$ denote the degree of the k^{th} vertex of the n^{th} level of a tree, then the sequence $d_{n,k}$ gives rise to a branching process in varying environments tree (**BPVET**) T^*. Let $d_{n,k}^+ = d_{n,k} - 1$. The random variables $d_{n,k}$ are assumed to be independent and for fixed n they are identically distributed. We consider the BPVET whose $d_{n,k}^+$'s are distributed as:

$d^+_{0,1} = d_{0,1} = 1$ a.s. and for $n \geq 1$,

$$d^+_{n,k} = \begin{cases} 1, & \text{with probability } 1 - q_n; \\ 2, & \text{with probability } q_n. \end{cases} \tag{1.9}$$

where $0 < q_n < 1$. Let Z^*_n denote the size of the n^{th} level S^*_n of T^*. For the rest of the paper, unit resistances are assigned to the edges of the trees.

The size of S^*_n is $Z^*_n = \sum_{k=1}^{Z^*_{n-1}} d^+_{n-1,k}$, $n \geq 2$ and $Z_1 = 1$ a.s. Obviously, $E(Z^*_n) = \prod_{i=1}^{n-1}(1 + q_i)$. The following theorem is a part of theorem 4.14 of [7].

Theorem 1. *If the sequence $\{d_{n,k}\}$ is locally bounded, then there exists $W^* > 0$ a.s. such that*

$$W^*_n = \frac{Z^*_n}{E(Z^*_n)} \to W^* \qquad a.s. \tag{1.10}$$

In which case, $Z^*_n \sim W^* E(Z^*_n)$ a.s.

The following theorem compares the expected values of the effective resistances of the SSRT and BPVET.

Theorem 2. [3] *Consider the two locally bounded, leafless trees, SSRT and BPVET such that for every $n \geq 0$ and $k \geq 1$, d_n and $d_{n,k}$ are identically distributed. Let R_{eff} and R^*_{eff} denote their respective effective resistances. Then*

$$E(R^*_{eff}) \leq E(R_{eff}) \tag{1.11}$$

It is worth mentioning that there is no stochastic domination between R_{eff} and R^*_{eff}, see [4].

1.2. **Auxiliary lemmas.**

The following useful lemmas are elementary and mostly straightforward.

Lemma 1. *Let $\{a_n\}$ and $\{b_n\}$ be two positive sequences such that $\lim_n \frac{a_n}{b_n} = L$.*

(i) *If $\sum_n b_n$ is divergent, then*

$$\lim_n \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} = L \tag{1.12}$$

(ii) *If $\sum_n b_n$ is convergent, then*

$$\lim_n \frac{\sum_{k=n}^\infty a_k}{\sum_{k=n}^\infty b_k} = L \tag{1.13}$$

Proof. (i) Let $a_n = t_n b_n$ such that $t_n \to L$ as $n \to \infty$. Given $\epsilon > 0$, then there exists n_0 sufficiently large such that for $n > n_0$,

$$(L - \epsilon)\sum_{k=1}^n b_k + \alpha \leq \sum_{k=1}^n a_k \leq (L + \epsilon)\sum_{k=1}^n b_k + \alpha'$$

where *alpha* and α' are constants. Since $\sum_n b_n$ is divergent then

$$L - \epsilon \leq \lim_n \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} \leq L + \epsilon$$

for arbitrary ϵ. Hence the result follows.

(ii) It follows directly using the same way of proving (i).

□

The following lemma follows using the same way of proving lemma 1, and that for $\beta > -1$,

$$\lim_n \frac{\sum_{k=1}^n k^\beta}{n^{\beta+1}} = \frac{1}{\beta+1} \tag{1.14}$$

while for $\beta < -1$,

$$\lim_n \frac{\sum_{k=n}^\infty k^\beta}{n^{\beta+1}} = \frac{-1}{\beta+1} > 0 \tag{1.15}$$

and

$$\lim_n \frac{\sum_{k=1}^n \frac{1}{k}}{\log n} = 1. \tag{1.16}$$

See [2, p.65].

Lemma 2. *Let $\{a_n\}$ be a positive sequence such that $\lim_n \frac{\log a_n}{\log n} = L < \infty$.*

(i) *If $L > -1$, then $\sum_n a_n$ is divergent and*

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = L + 1. \tag{1.17}$$

(ii) *If $L < -1$, then $\sum_n a_n$ is convergent and*

$$\lim_n \frac{\log \sum_{k=n}^\infty a_k}{\log n} = L + 1. \tag{1.18}$$

(iii) *If $L = -1$, then*

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = 0. \tag{1.19}$$

Lemma 3. *Let $\{a_n\}$ be a positive sequence.*

(i) *If $\lim_n \frac{\log a_n}{\log n} = \infty$, then $\sum_n a_n$ is divergent and*

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = \infty. \tag{1.20}$$

(ii) *If $\lim_n \frac{\log a_n}{\log n} = -\infty$, then $\sum_n a_n$ is convergent and*

$$\lim_n \frac{\log \sum_{k=n}^\infty a_k}{\log n} = -\infty. \tag{1.21}$$

Lemma 4. *Let* $\{a_n\}$ *be a sequence of real numbers such that* $\lim_n na_n$ *exists. Then,*

(i) *If* $a_n > 0$*, then* $\lim_n na_n = \lim_n \frac{\sum_{k=1}^n a_k}{\log n}$.

(ii) *If* $a_n > -1$ *and* $a_n \to 0$*, then* $\lim_n na_n = \lim_n \frac{\sum_{k=1}^n \log(1+a_k)}{\log n}$.

Proof. (i) It is direct using lemma 1 and equation (1.16).

(ii) Since $\lim_n a_n = 0$, then $\lim_n (1+a_n)^n = e^{\lim_n na_n}$ and the rest follows using part (i). □

2. Resistance dimension

In [5], we have obtained the following result, assuming that $\lim_n nq_n$ exists.

Theorem 3. [5] *Consider a SSRT whose* d_n^+*'s are defined by (1.8), such that* $q_n \to 0$ *as* $n \to \infty$*. Then* $d_\Omega = 1 + \log 2 \lim_n nq_n$ *a.s.*

It is easy to generalize the last theorem for locally bounded, leafless SSRT's.

In the following theorem we study the resistance dimension of the BPVET. The trees that we consider are leafless and for this reason the limits involved in the definition of the resistance dimension do exist and we would not bother with the lim sup that appears in that definition.

Theorem 4. *Consider the BPVET whose* $d_{n,k}^+$*'s are given by (1.9) such that* $q_n \to 0$ *as* $n \to \infty$*. Then*

$$1 + \frac{1}{2} \lim_n nq_n \leq E(d_\Omega^*) \leq 1 + \lim_n nq_n \tag{2.1}$$

provided that $\lim_n nq_n$ *exists and in the case of transient random walk,* $\lim_n nq_n > 2$ *is assumed.*

Proof. We first consider the case of recurrence. It follows from inequality (1.6) that

$$d_\Omega^* = 2 - \lim_n \frac{\log R_n^*}{\log n} \leq 2 - \lim_n \frac{\log \sum_{k=1}^n \frac{1}{Z_k^*}}{\log n}$$

It follows from theorem 1 and lemma 4 that

$$\begin{aligned} \lim_n \frac{\log Z_n^*}{\log n} &= \lim_n \frac{\sum_{k=1}^{n-1} \log(1+q_k)}{\log n} \\ &= \lim_n nq_n \qquad a.s. \end{aligned}$$

Hence, lemmas 2 and 3 imply that $d_\Omega^* \leq 1 + \lim_n nq_n$ a.s.

On the other hand,

$$\begin{aligned} E(d^*_\Omega) &= 2 - E(\lim_n \tfrac{\log R^*_n}{\log n}) \\ &\geq 2 - \lim_n E(\tfrac{\log R^*_n}{\log n}) \\ &\geq 2 - \lim_n \tfrac{\log E(R^*_n)}{\log n} \\ &\geq 2 - \lim_n \tfrac{\log E(R_n)}{\log n} \end{aligned}$$

where the first inequality follows from Fatou's lemma and the second follows from Jensen's inequality, while the third one follows from theorem 2. Since

$$E(\frac{1}{Z_n}) = \prod_{k=1}^{n-1}(1 - \frac{1}{2}q_k)$$

then it follows from lemma 4 that

$$\begin{aligned} \lim_n \tfrac{\log E(\frac{1}{Z_n})}{\log n} &= \lim_n \tfrac{\sum_{k=1}^{n-1} \log(1-\frac{1}{2}q_k)}{\log n} \\ &= -\tfrac{1}{2}\lim_n nq_n \qquad a.s. \end{aligned} \tag{2.2}$$

Since for SSRT, $R_n = \sum_{k=1}^{n} \frac{1}{Z_k}$ a.s., it follows then from lemma 2 that

$$\lim_n \frac{\log E(R_n)}{\log n} = \lim_n \frac{\log \sum_{k=1}^{n} E(\frac{1}{Z_k})}{\log n} = 1 - \frac{1}{2}\lim_n nq_n$$

Thus,

$$E(d^*_\Omega) \geq 1 + \frac{1}{2}\lim_n nq_n$$

and this completes the proof for the recurrence case.

We now consider the case of transience for which,

$$\begin{aligned} d^*_\Omega &= 2 - \lim_n \tfrac{\log(R^* - R^*_n)}{\log n} \\ &\leq 2 - \lim_n \tfrac{\log \sum_{k=n+1}^{\infty} \frac{1}{Z^*_k}}{\log n} \\ &= 1 + \lim_n \tfrac{\log Z^*_n}{\log n} \qquad a.s. \end{aligned}$$

where the last equality is due to lemma 2. Theorem 1 assures that

$$\begin{aligned} d^*_\Omega &\leq 1 + \lim_n \tfrac{\log E(Z^*_n)}{\log n} \\ &= 1 + \lim_n \tfrac{\sum_{k=1}^{n-1} \log(1+q_k)}{\log n} \\ &= 1 + \lim_n nq_n \qquad a.s. \end{aligned}$$

where the last equality is due to lemma 4. On the other hand,

$$\begin{aligned} E(d^*_\Omega) &= 2 - E(\lim_n \tfrac{\log(R^* - R^*_n)}{\log n}) \\ &\geq 2 - \lim_n E(\tfrac{\log(R^* - R^*_n)}{\log n}) \\ &\geq 2 - \lim_n \tfrac{\log E(R^* - R^*_n)}{\log n} \end{aligned}$$

Using theorem 2 and then lemma 2, we obtain

$$\begin{aligned} E(d^*_\Omega) &\geq 2 - \lim_n \tfrac{\log E(R - R_n)}{\log n} \\ &= 2 - \lim_n \tfrac{\log \sum_{k=n+1}^{\infty} E(\frac{1}{Z_k})}{\log n} \\ &= 1 - \lim_n \tfrac{\log E(\frac{1}{Z_n})}{\log n} \end{aligned}$$

and equation (2.2) implies that

$$E(d^*_\Omega) \geq 1 + \frac{1}{2} \lim_n nq_n$$

which completes the proof of the theorem. □

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