THE LIMITING SPECTRA OF RANDOM BLOCK-MATRICES

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Abstract

Random block-matrices are of a particular interest in some branches of science and some applications, e.g., physics, statistics, time series, image processing and wireless communication. In my research, I study the limiting spectral distribution of some patterns of random block-matrices. Interestingly, they turn out to be mixtures of probability distributions. In Oraby (2005), I analyze among other things the convergence of random block-matrices. In a recent joint paper Far et al. (2006) we use matrix-valued Cauchy transforms to compute the limiting spectra of certain block-matrices. This poster, which presents my results in Oraby (2006), describes a model for random block-matrices which generalizes a setting considered by Girko (2000). In this poster, I am going to present its limiting spectral distribution under general conditions, outline the proof, give some examples for this model and discuss some other cases.

Introduction

A $k \times k$ random block-matrix $\mathbb{B}$ is a matrix whose entries are random matrices. If the entries of a Hermitian block- matrix $\mathbb{B}$ are $n \times n$ matrices then $\mathbb{B}$ is an $nk \times nk$ Hermitian matrix. Random tri-diagonal block-matrices appear as transport matrices in physics c.f. Molinari (2003). In wireless communications, random circulant block-matrices appear as channel matrices c.f. Muller (2004)

Girko (Girko 1998, p.100) introduces a system of equations (called the canonical equations) for random block-matrices in both cases of independent and asymptotically independent blocks. The resolvent of the random block-matrix follows from the solution of the canonical equations. He also examines in Girko (2000) his procedure for some specific random block-matrices. Nevertheless, the limiting spectral distribution was not identified. In Oraby (2006) we identified the limiting spectral distribution for the pattern considered in Girko (2000) under more general conditions.

Preliminaries and Definitions

Table 1. Notations and definitions.

By ...	we mean ...
$\mu_{\mathbf{A}} = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}$	the spectral measure of an $n \times n$ Hermitian matrix $\mathbf{A}$ with the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.
$\mathrm{tr}_n(\mathbf{A})$	$\frac{1}{n}\sum_{i=1}^{n} A_{ii}$
$\lim_{n\to\infty}\mu_n \overset{\mathcal{D}}{=} \mu$ a.s. or $\mu_n \xrightarrow{\mathcal{D}} \mu$ as $n \to \infty$ $\quad a.s.$	the weak convergence of a sequence of random measures $\{\mu_n\}$ to a measure μ with probability one.
$\mu_n \xrightarrow{\mathrm{m}} \mu$ as $n \to \infty$ $\quad a.s.$	the convergence in moments of a sequence of random probability measures $\{\mu_n\}$ to μ with probability one.
$\mathbf{A} = (A_{ij})$ is an $n \times n$ Wigner matrix	an $n \times n$ matrix for which $A_{ij} = \overline{A}_{ji}$ are i. i. d. complex r. v. for all $i < j$ such that $E(A_{ij}) = 0$ and $E(\lvert A_{ij}\rvert^2) = \frac{\sigma^2}{n} < \infty$
$\mathbf{Wigner}(n, \sigma^2)$	the space of all such $n \times n$ Wigner matrices
$\mathbf{A} = (A_{ij})$ is an $n \times n$ Gaussian matrix	an $n \times n$ Wigner matrix such that $A_{ii} \sim N(0, \frac{\sigma^2}{n})$ and $\Re A_{ij}, \Im A_{ij} \sim N(0, \frac{\sigma^2}{2n})$ for every $i < j$
$\mathbf{Gaussian}(n, \sigma^2)$	the space of all such $n \times n$ Gaussian matrices
$\mathbf{B} = \mathbf{X}^T\mathbf{X}$ is an $n \times n$ Wishart matrix	if the random matrix $\mathbf{X} = (X_{ij})$ is a $p \times n$ random matrix and $\Re X_{ij}, \Im X_{ij} \sim N(0, \frac{\sigma^2}{2n})$ are independent for every i, j
$\mathbf{Wishart}(p, n, \sigma^2)$	the space of all such $n \times n$ Wishart matrices
γ_{α,σ^2}	the semicircular law centered at α and of variance σ^2 given as $\gamma_{\alpha,\sigma^2}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-\alpha)^2}\ \mathbf{1}(\lvert x - \alpha \rvert \le 2\sigma)dx$.

Our Model

Consider the sequence $\{\mathbb{B}_{n,k}\}$ of the random block-matrices:

$$\mathbb{B}_{\mathbf{n},\mathbf{k}} = \begin{bmatrix} \mathbf{A}_n + w_{11}\mathbf{B}_n & w_{12}\mathbf{B}_n & \dots & w_{1k}\mathbf{B}_n \\ w_{21}\mathbf{B}_n & \mathbf{A}_n + w_{22}\mathbf{B}_n & \dots & w_{2k}\mathbf{B}_n \\ \vdots & \vdots & \ddots & \vdots \\ w_{k1}\mathbf{B}_n & w_{k2}\mathbf{B}_n & \dots & \mathbf{A}_n + w_{kk}\mathbf{B}_n \end{bmatrix}$$

where

- $\mathbf{A}_n$ and $\mathbf{B}_n$ are two $n \times n$ random Hermitian matrices
- $\{w_{ij} : 1 \le i, j\}$ is a family of integrable random variables.

Simulations

We carried out simulations (using Matlab) for the eigenvalues of the random block-matrix $\mathbb{B}_{n,k}$ for $k = n = 40$.

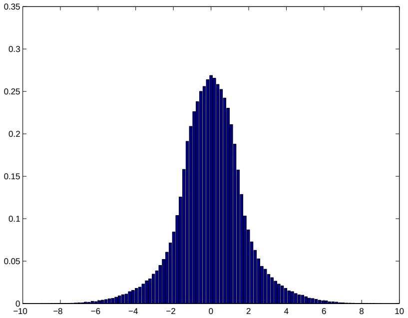

Figure 1. Histogram of the eigenvalues of $\mathbb{B}$ when $\mathbf{A} \in \mathbf{Gaussian}(40,1)$ and $\mathbf{B} \in \mathbf{Wishart}(40,40,1)$.

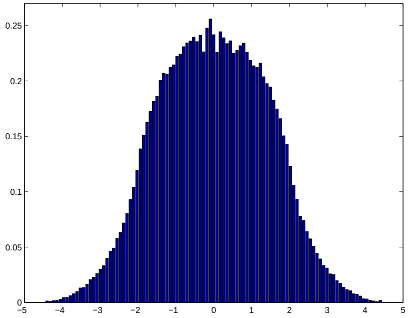

Figure 2. Histogram of the eigenvalues of $\mathbb{B}$ when $\mathbf{A}$ and $\mathbf{B} \in \mathbf{Gaussian}(40,1)$.

Assumptions

It is clear that the double array sequence of random block-matrices $\{\mathbb{B}_{n,k}\}$ has terms read as

$$\mathbb{B}_{n,k} = \mathbf{I}_k \otimes \mathbf{A}_n + \mathbf{W}_k \otimes \mathbf{B}_n$$

where

$$\mathbf{W}_k = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1k} \\ w_{21} & w_{22} & \dots & w_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{k1} & w_{k2} & \dots & w_{kk} \end{bmatrix}$$

$\mathbf{I}_k$ is the $k \times k$ identity matrix and $\otimes$ is the Kronecker product. Assume that:

(i) $\{\mathbf{W}_k\}$ is a sequence of Hermitian random matrices for which

$$\mu_{\mathbf{W}_k} \xrightarrow{\mathrm{m}} \rho \ \text{ as } k \to \infty \qquad a.s.$$

such that ρ is a non-random probability measure with the compact support $[t_1, t_2]$.

(ii) $\{\mathbf{A}_n\}$ and $\{\mathbf{B}_n\}$ are two sequences of Hermitian random matrices independent of k and $\{\mathbf{W}_k\}$ and

$$\mu_{\mathbf{A}_n + t\mathbf{B}_n} \xrightarrow{\mathrm{m}} \phi(t;.) \ \text{ as } n \to \infty \qquad a.s.$$

for all $t \in [t_1, t_2]$, where $\phi(t;.)$ has a compact support that is uniformly bounded in $t \in [t_1, t_2]$.

The Result

Theorem 1. Oraby (2006). Under the above assumptions:

$$\lim_{\mathbf{n}\to\infty}\lim_{\mathbf{k}\to\infty}\mu_{\mathbb{B}_{\mathbf{n},\mathbf{k}}} = \lim_{\mathbf{k}\to\infty}\lim_{\mathbf{n}\to\infty}\mu_{\mathbb{B}_{\mathbf{n},\mathbf{k}}} \overset{\mathcal{D}}{=} \nu \qquad \mathbf{a.s.}$$

where ν has a bounded support and is defined as

$$\nu(\mathbf{dx}) = \int_{\mathbf{t_1}}^{\mathbf{t_2}} \phi(\mathbf{t};\mathbf{dx})\,\rho(\mathbf{dt})$$

An Outline of The Proof

(i) The proof in Oraby (2006) is based on the method of moments.

(ii) Using some properties of the Kronecker products, it follows that:

$$\mathrm{tr}_{nk}(\mathbb{B}_{n,k}^m) = \int_{\mathbb{R}} \mathrm{tr}_n((\mathbf{A}_n + t\mathbf{B}_n)^m)\mu_{\mathbf{W}_k}(dt)$$

(iii) It follows also that:

$$\mathrm{tr}_{nk}(\mathbb{B}_{n,k}^m) = \int_{\mathbb{R}}\int_{\mathbb{R}} x^m \mu_{\mathbf{A}_n + t\mathbf{B}_n}(dx)\mu_{\mathbf{W}_k}(dt)$$

(iv) The two iterated limits then follow from the convergence in moments and the observation that $\mathrm{tr}_n((\mathbf{A}_n + t\mathbf{B}_n)^m)$ converges to a polynomial of degree m in t.

Examples

Example 1. Suppose that

- $\mathbf{W}_k \in \mathbf{Wigner}(k,1)$ for all k,
- $\mathbf{A}_n \in \mathbf{Gaussian}(n,1)$ for all n,
- $\mathbf{B}_n \in \mathbf{Wishart}(p,n,1)$ such that $\lim_{n\to\infty}\frac{p}{n} = 1$ and which is independent of $\mathbf{A}_n$ for all n.

Then ν has a probability density function given as

$$\mathbf{g}(\mathbf{x}) = \int_{\mathbb{R}} \mathbf{f}(\mathbf{x};\mathbf{t})\,\gamma_{\mathbf{0},\mathbf{1}}(\mathbf{dt})$$

where $f(x;t)$ is given by

$$f(x;t) = \frac{1}{2\sqrt{3}\,\pi\,t}\left(H(x;t) - \frac{h_2(x;t)}{H(x;t)}\right)\mathbf{1}(s_1(t) < x < s_2(t))$$

where

$$H(x;t) = \frac{1}{\sqrt[3]{2}}\left(h_1(x;t) + \sqrt{h_1(x;t)^2 - 4\,h_2(x;t)^3}\right)^{\frac{1}{3}},$$
$$h_1(x;t) = 2 + 27\,t^2 - 3\,t\,x - 3\,t^2\,x^2 + 2\,t^3\,x^3$$

and

$$h_2(x;t) = 1 - t\,x + t^2\,x^2$$

and $s_1(t), s_2(t)$ are uniquely determined.

Example 2. Suppose that

- $\mathbf{W}_k \in \mathbf{Wigner}(k,1)$ for all k,
- $\mathbf{A}_n$ and $\mathbf{B}_n \in \mathbf{Gaussian}(n,1)$ for all n and are independent.

Then ν has a probability density function given as

$$g(x) = \begin{cases} g_1(x) & ;\ \text{whenever } 2 \le \lvert x\rvert \le 2\sqrt{5} \\ g_2(x) & ;\ \text{whenever } \lvert x\rvert \le 2 \end{cases}$$

where

$$\mathbf{g_1}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}\pi^{\mathbf{2}}}\int_{\sqrt{\frac{\mathbf{x}^2}{4}-\mathbf{1}}}^{\mathbf{2}} \frac{\sqrt{\mathbf{4}(\mathbf{1}+\mathbf{t}^{\mathbf{2}}) - \mathbf{x}^{\mathbf{2}}}\,\sqrt{\mathbf{4}-\mathbf{t}^{\mathbf{2}}}}{(\mathbf{1}+\mathbf{t}^{\mathbf{2}})}\,\mathbf{dt},$$

and

$$\mathbf{g_2}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}\pi^{\mathbf{2}}}\int_{\mathbf{0}}^{\mathbf{2}} \frac{\sqrt{\mathbf{4}(\mathbf{1}+\mathbf{t}^{\mathbf{2}}) - \mathbf{x}^{\mathbf{2}}}\,\sqrt{\mathbf{4}-\mathbf{t}^{\mathbf{2}}}}{(\mathbf{1}+\mathbf{t}^{\mathbf{2}})}\,\mathbf{dt}.$$

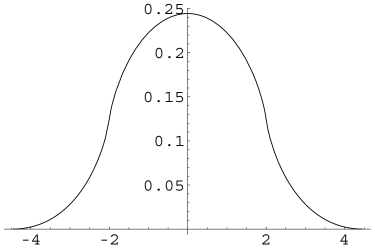

Figure 3. The probability density function corresponding to the limiting spectral distribution of $\mathbb{B}$ when $\mathbf{A}$ and $\mathbf{B} \in \mathbf{Gaussian}(n,1)$. Compare it with Figure 2.

Another Case

Proposition 2. Oraby (2006). For a fixed finite $k \ge 1$:

$$\lim_{\mathbf{n}\to\infty}\mu_{\mathbb{B}_{\mathbf{n},\mathbf{k}}} \overset{\mathcal{D}}{=} \nu_{\mathbf{k}} \qquad \mathbf{a.s.}$$

where ν_k has a bounded support and is defined as

$$\nu_{\mathbf{k}}(\mathbf{dx}) = \int_{\mathbb{R}} \phi(\mathbf{t};\mathbf{dx})\,\mu_{\mathbf{W}_{\mathbf{k}}}(\mathbf{dt})$$

This follows from the proof of Theorem 1 (see the outline of the proof (iii) above.)

Examples

Example 3. Suppose that $w_{ij} = 1 - \mathbf{1}_{i=j}$ for all $1 \le i, j \le k$ then by induction $\mu_{\mathbf{W}_k} = \frac{k-1}{k}\delta_{-1} + \frac{1}{k}\delta_{k-1}$ and so

$$\mu_{\mathbb{B}_{\mathbf{n},\mathbf{k}}} \xrightarrow{\mathcal{D}} \frac{\mathbf{k}-\mathbf{1}}{\mathbf{k}}\,\phi(-\mathbf{1},.) + \frac{\mathbf{1}}{\mathbf{k}}\,\phi(\mathbf{k}-\mathbf{1},.) \ \text{ as } n \to \infty \qquad \mathbf{a.s.}$$

Example 4. Suppose that $w_{ij} = \mathbf{1}_{j=i\pm1}$ for all $1 \le i, j \le k$ then

$$\mu_{\mathbf{W}_k} = \frac{1}{k}\sum_{r=1}^{k}\delta_{2\cos(\frac{r\pi}{k+1})}$$

and so

$$\mu_{\mathbb{B}_{\mathbf{n},\mathbf{k}}} \xrightarrow{\mathcal{D}} \frac{\mathbf{1}}{\mathbf{k}}\sum_{\mathbf{r}=\mathbf{1}}^{\mathbf{k}}\phi(\mathbf{2}\cos(\frac{\mathbf{r}\pi}{\mathbf{k}+\mathbf{1}}),.) \ \text{ as } n \to \infty \qquad \mathbf{a.s.}$$

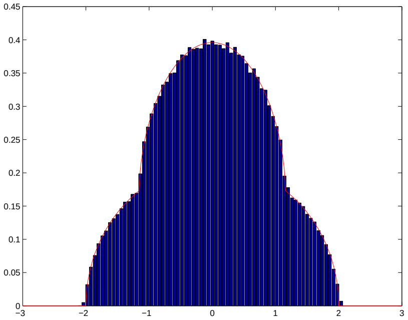

Figure 4. The simulation and the exact probability density function of the limiting spectral distribution of $\mathbb{B}$ discussed in Example 4 with $k = 3$ and $\mathbf{A}$ and $\mathbf{B} \in \mathbf{Gaussian}(n,1)$.

One Further Case

Theorem 3. Oraby (2006). Suppose that $\mathbf{W}_k \in \mathbf{Wigner}(k,1)$ for all k. Let

$$\mathbf{A} = \mathbf{U}\,\mathbf{Diag}(\alpha_1,\dots,\alpha_n)\,\mathbf{U}^* \text{ and } \mathbf{B} = \mathbf{U}\,\mathbf{Diag}(\beta_1,\dots,\beta_n)\,\mathbf{U}^*$$

be two $n \times n$ Hermitian random matrices with the eigenvalues $\{\alpha_1,\cdots,\alpha_n\}$ and $\{\beta_1,\cdots,\beta_n\}$, respectively, where $\mathbf{U}$ is a random matrix of order n with the uniform distribution on the unitary group $\mathbb{U}(n)$. Then

$$\mu_{\mathbb{B}_{\mathbf{n},\mathbf{k}}} \xrightarrow{\mathrm{m}} \frac{\mathbf{1}}{\mathbf{n}}\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}\gamma_{\alpha_{\mathbf{j}},\beta_{\mathbf{j}}^{\mathbf{2}}} \ \text{ as } k \to \infty \qquad \mathbf{a.s.}$$

If the α's and β's have some finite exponential moments then the weak convergence follows.

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REFERENCES

Far, R., Oraby, T., Bryc, W., and Speicher, R. (2006). Spectra of large block matrices. *Submitted.*

Girko, V. L. (1998). *An introduction to statistical analysis of random arrays.* VSP, Utrecht. Translated from the Russian, Revised by the author.

Girko, V. L. (2000). Random block matrix density and *SS*-law. *Random Oper. Stochastic Equations*, 8(2):189–194.

Molinari, L. (2003). Spectral duality and distribution of exponents for transfer matrices of block-tridiagonal Hamiltonians. *J. Phys. A*, 36(14):4081–4090.

Muller, R. (2004). Random matrices, free probability and the replica method. *European Signal Processing Conference (Eusipco), Vienna, Austria.* http://userver.ftw.at/ mueller/papers/eusipco04.pdf.

Oraby, T. (2005). The spectral laws of symmetric block-matrices with large random blocks. *Submitted.*

Oraby, T. (2006). The limiting spectra of girko's block-matrix. *Submitted.*