Fractal Dimensions and Random Walks on Random Trees

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Abstract. We determine the fractal dimension d_f of infinite spherically symmetric random trees (all vertices at distance n from the root have the same degree d_n where $\{d_n\}$ are independent random variables). If d_n takes the value 3 or 2 with probabilities q_n and $1-q_n$, then $d_f = (\log 2)\lim_n nq_n + 1$ *a.s.* We show how d_f is closely related to the type of the simple random walks (SRW) on trees. We prove that the SRW is *a.s.* transient if $d_f > 2$ *a.s.* and *a.s.* recurrent if $d_f < 2$ *a.s.* and if $d_f = 2$ *a.s.* we obtain *a.s.* transience or recurrence. We also consider another type of random trees which are corresponding to branching processes in varying environments (BPVE). In particular, we consider a tree such that the degrees of the vertices at distance n from the root are independent identically distributed (iid) random variables following the distribution of a random variable that takes the value 3 or 2 with probabilities q_n and $1-q_n$ respectively. These iid random variables are also independent of the degrees of the vertices of the other generations. We prove that the SRW is *a.s.* recurrent if and only if $d_f \leq 2$ *a.s.* We also prove for such trees that $d_f = \lim_n nq_n + 1$ *a.s.* Some of the results of this paper sharpens some of those of [6].

1 Introduction.

We study in this paper the fractal dimensions of random trees and show how they are closely related to the type of SRW on those trees. This paper is motivated by the paper of Telcs [11]. By a tree T we mean a connected graph with no loops or cycles and a vertex r is distinguished to be the root. In addition, it is assumed to be leafless (every vertex except possibly the root has degree at least 2) and has a countably infinite set of vertices. By a distance between two vertices we mean the number of edges of the shortest path connecting them. Let Z_n stands for the size of the n^{th} level i.e. the number of vertices at distance n from the root. The fractal dimension of a tree T is defined to be

$$d_f = \lim_n \frac{\log \sum_{k=1}^n Z_k}{\log n}$$

The degree $d(x)$ of a vertex x is defined to be the number of vertices adjacent to x. Obviously, d_f equals 1 for the homogeneous tree of degree 2 and equals infinity for the homogeneous tree of degree 3. Not surprisingly, d_f for a tree of mixed degrees 2 or 3 can be finite or infinite. The fractal dimension of the Galton-Watson tree in which every vertex has degree 3 or 2 with probabilities q and $1-q$, respectively, is proved in section 2 to be infinity. Therefore a tree of mixed degrees may attain a finite d_f if the degree 3 vertices become sparser as the tree grows. In order to explore the phase transition between finite and infinite d_f we chose to study types of random trees in which the probability of degree 3 vertex is not fixed, as in the case of Galton-Watson Trees, but rather converges to 0 fast enough as its distance from the root tends to infinity. In section 3, we consider the spherically symmetric random trees, whereas branching processes in varying environments trees (BPVET) will be considered in section 4. The type of the random walk (transient - recurrent) on random trees has been approached in a similar way. See [2], [8], or [9]. The SRW on a tree is a Markov chain with the set of vertices as the state space and the one-step transition probabilities are defined by:

$$P(x,y) = \begin{cases} \frac{1}{d(x)} & \text{if } y \text{ is adjacent to } x, \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the SRW on the homogeneous tree of degree 2 is recurrent and transient on the homogeneous tree of degree 3. See [2] or [8]. As it has been expected, the SRW on the Galton-Watson tree described above is a.s. transient. The type of the SRW on spherically symmetric trees as well as on BPVE trees is also studied and the critical trees is entirely settled for the former in [8] and in [9] for the later trees. This will be exposed in sections 3 and 4, respectively.

2 Galton-Watson Trees.

Consider a random variable X with a probability function $p(X=k) = p_k$, $k = 0, 1, 2, \ldots$ and define a sequence $\{Z_n\}$ of random variables in the way that

$$Z_0 = 1 \text{ and for } n \geq 1, \; Z_n = \sum_{i=1}^{Z_{n-1}} d^+(i) \tag{1}$$

where $d^+(i) = d(i) - 1$ are iid random variables following the same distribution of X. This is a type of family trees in which every individual i produces independently of the other individuals a number $d^+(i)$ of offspring according to a certain probability distribution that is independent of i. Z_n in (1) stands for the size of the n^{th} level. We consider a particular example of Galton Watson trees in which the offspring of any individual follows the distribution (2). For $0 < q < 1$, let

$$d^+(i) = \begin{cases} 1 & \text{with probability } 1-q \\ \\ 2 & \text{with probability } q \end{cases} \tag{2}$$

It is known that $W_n = \frac{Z_n}{E(Z_n)}$ is a nonnegative martingale and then it has a finite a.s. limit W. Since $E(d^+(i)) = 1 + q > 1$, then this martingale is supercritical and the limit $W > 0$ a.s. See [1, p.9]. Using that $E(Z_n) = (1+q)^n$ and lemma 2 below we get

$$\lim_n \frac{\sum_{k=1}^n Z_k}{\sum_{k=1}^n (1+q)^k} = W \quad a.s.$$

Now,

$$b_n = \sum_{k=1}^n Z_k = t_n W \sum_{k=1}^n (1+q)^k; \quad t_n \to 1 \quad a.s.$$

Whence,

$$d_f = \lim_n \frac{\log b_n}{\log n} = \infty \quad a.s.$$

More generally, $d_f = \infty$ for all supercritical non-extinct Galton-Watson trees.

3 Spherically Symmetric trees.

In this section we consider the type of spherically symmetric trees in which the degree of every vertex depends only on its distance from the root r. Let $d_n^+ = d_n - 1$ where d_n denotes the degree of every vertex at distance n from r and Z_n stands for the size of the n^{th} level. Also $\{d_n\}$ are assumed to be independent random variables. Whence,

$$Z_n = \prod_{k=0}^{n-1} d_k^+$$

and then

$$\log Z_n = \sum_{k=0}^{n-1} \log d_k^+$$

is a sum of independent random variables.
The following lemma plays an essential role in this section.

Lemma 1. *[8]*
Suppose that X_n, $n = 1, 2, \dots$ are independent random variables with $0 \le X_n \le M$ for some constant M. Let

$$S_n = \sum_{k=1}^{n} X_k$$

If $E(S_n) \to \infty$ as $n \to \infty$, then

$$\frac{S_n}{E(S_n)} \longrightarrow 1 \quad a.s.$$

The following two elementary lemmas are also needed.

Lemma 2. *[4, p.34]*
Let $\{a_n\}$ and $\{b_n\}$ be two positive sequences such that $\sum_n b_n$ is divergent. If $\lim_n \frac{a_n}{b_n} = L$, then

$$\lim_n \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} = L$$

Lemma 3. *If the nonnegative sequence $\{a_n\}$ is non-increasing, then*
(1) $\lim_n n\, a_n = \lim_n \frac{\sum_{k=1}^n a_k}{\log n}$.
(2) $\lim_n n\, a_n = \lim_n \frac{\sum_{k=1}^n \log(1+a_k)}{\log n}$, *provided that $a_n \to 0$ as $n \to \infty$.*

Proof. (1) It follows from lemma 2 that

$$\lim_n \frac{\sum_{k=1}^n a_k}{\log n} = \lim_n \frac{a_n}{1/n} = \lim_n n\, a_n$$

(2) From part (1),

$$\begin{aligned} \lim_n \tfrac{\sum_{k=1}^n \log(1+a_k)}{\log n} &= \lim_n n \log(1+a_n) \\ &= \lim_n \log(1+\tfrac{n\, a_n}{n})^n \\ &= \log \exp(\lim_n n\, a_n) = \lim_n n\, a_n \end{aligned}$$

□

We now consider random spherically symmetric trees where the degree sequence $\{d_n\}$ is such that

$$d_n^+ = \begin{cases} 1 & \text{with probability } 1 - q_n \\ \\ 2 & \text{with probability } q_n \end{cases} \tag{3}$$

where $0 < q_n < 1$. But first we introduce the following lemma.

Lemma 4. *Consider a positive sequence $\{a_n\}$. If $\lim_n \frac{\log a_n}{\log n} = L$ for some $L \geq 0$, then*

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = L + 1$$

Proof. For the case $L = \infty$, we have nothing to prove. So, we assume that $0 \leq L < \infty$. We first consider the case $L > 0$. If $\lim_n \frac{\log a_n}{\log n} = L$, then

$$a_n = n^{Lt_n}; \quad t_n \to 1 \text{ as } n \to \infty$$

For arbitrary small $\epsilon > 0$ and sufficiently large n, $L - \epsilon \leq Lt_n \leq L + \epsilon$ and then for some constants A and A',

$$A + \sum_{k=1}^n k^{L-\epsilon} \leq \sum_{k=1}^n k^{Lt_k} \leq A' + \sum_{k=1}^n k^{L+\epsilon}$$

Consequently,

$$\lim_n \frac{\log \sum_{k=1}^n k^{L-\epsilon}}{\log n} \leq \lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = \lim_n \frac{\log \sum_{k=1}^n k^{Lt_k}}{\log n} \leq \lim_n \frac{\log \sum_{k=1}^n k^{L+\epsilon}}{\log n}$$

Choosing ϵ such that $L - \epsilon > 0$ and using the fact that for $\beta \geq 0$,

$$\lim_n \frac{\sum_{k=1}^n k^\beta}{n^{\beta+1}} = \frac{1}{\beta + 1}$$

we obtain,

$$L + 1 - \epsilon \leq \lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} \leq L + 1 + \epsilon$$

Whence,

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = L + 1.$$

For the case $L = 0$, we obtain

$$1 - \epsilon \leq \lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} \leq 1 + \epsilon$$

where the right hand side inequality is obtained using the same argument above and the left hand side inequality is obtained using the fact that for $0 < \gamma \leq 1$,

$$\sum_{k=1}^n k^{-\gamma} \geq \frac{(n+1)^{1-\gamma} - 1}{1 - \gamma}$$

□

Note: The argument for the case $L = 0$ extends for any value $-1 \leq L \leq 0$. Whereas, if $L < -1$, the series $\sum_n a_n$ converges and the limit

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = 0.$$

The following theorem gives a criterion to determine the fractal dimension of the spherically symmetric trees.

Theorem 5. *If $\{d_n\}$ is as defined in (3), then*

$$d_f = 1 + (\log 2) \lim_n n\, q_n \quad a.s. \tag{4}$$

Proof. If $\sum_n q_n < \infty$, then $\lim_n n\, q_n = 0$ and the right hand side of equation (4) equals 1. On the other hand, if $\sum_n q_n = \sum_n p(d_n^+ = 2) < \infty$, it follows from Borel-Cantelli lemma that $p(d_n^+ = 2 \ i.o.) = 0$. Hence, $d_n^+ = 1$ eventually a.s. and, as such, there is exists a random variable $Z < \infty$ a.s. such that $Z_n \leq Z$ a.s. Whence,

$$\lim_n \frac{\log \sum_{k=1}^n Z_k}{\log n} \leq \lim_n \frac{\log(n\, Z)}{\log n} = 1 \quad a.s.$$

But, it follows from the definition of d_f that $d_f \geq 1$. Thus $d_f = 1$ and equation (4) holds true. Suppose now that $\sum_n q_n = \infty$. Obviously,

$$E(\log Z_n) = \sum_{k=0}^{n-1} E(\log d_k^+) = (\log 2) \sum_{k=0}^{n-1} q_k$$

It follows, as a consequence of lemma 1, that

$$\frac{\log Z_n}{E(\log Z_n)} \longrightarrow 1 \quad a.s.$$

and consequently,

$$\begin{aligned}\lim_n \tfrac{\log Z_n}{\log n} &= \lim_n \tfrac{\log Z_n}{E(\log Z_n)} \cdot \tfrac{E(\log Z_n)}{\log n} \\ &= \lim_n \tfrac{E(\log Z_n)}{\log n} = \lim_n \tfrac{(\log 2)\sum_{k=0}^{n-1} q_k}{\log n} \quad a.s.\end{aligned}$$

Lemma 3 assures that

$$\lim_n \frac{\log Z_n}{\log n} = (\log 2) \lim_n n\, q_n \quad a.s.$$

and lemma 4 completes the proof. □

The following theorem explores the phase transition between recurrence and transience of the SRW on spherically symmetric trees of mixed degrees in terms of the value of the fractal dimension. The Nash-Williams' theorem [10] plays a crucial role in exposing the relationship between the effective resistance of an electric network and the random walk on the underlying graph. It asserts that the SRW on a graph G is transient if and only if the effective resistance of G, when a one-unit resistance is assigned to every edge of G, is finite. See also [2]. The following lemma is extracted from Nash-Williams' theorem.

Lemma 6. *The SRW on a spherically symmetric tree T is transient if and only if*

$$\sum_n \frac{1}{Z_n} < \infty. \tag{5}$$

Theorem 7. *If the degree sequence is as defined in (3), then the SRW on T is*
(1) a.s. recurrent if $d_f < 2$ a.s.
(2) a.s. transient if $d_f > 2$ a.s.

Proof. (1) It follows from Cauchy-Schwartz inequality that

$$\sum_{k=1}^n Z_k \sum_{k=1}^n \frac{1}{Z_k} \geq n^2$$

Suppose that $d_f = d$. From the definition of d_f, we obtain

$$b_n = \sum_{k=1}^n Z_k = n^{dt_n}; \quad t_n \to 1 \quad a.s. \tag{6}$$

Whence,

$$\sum_{k=1}^n \frac{1}{Z_k} \geq n^{2-dt_n}$$

Now if $d < 2$ a.s., then

$$\lim_n \sum_{k=1}^{n} \frac{1}{Z_k} = \infty \quad a.s.$$

and the recurrence follows from (5).
(2) On the other hand, it follows from (6) that

$$b_n = n^{dt_n} \leq n\, Z_n$$

and hence,

$$\frac{1}{Z_n} \leq \frac{1}{n^{dt_n - 1}}$$

If $d > 2$ a.s., then

$$\lim_n \sum_{k=1}^{n} \frac{1}{Z_k} \leq \lim_n \sum_{k=1}^{n} \frac{1}{k^{dt_k - 1}} < \infty \quad a.s.$$

and the transience follows from (5). □

Therefore, the value $d_f = 2$ represents the critical value. The following example of [7] shows that we get both transience and recurrence when $d_f = 2$. Corollary 8 below has been proven in [8] using an argument similar to that given here and in [5] using the law of iterated logarithm. This corollary follows also by combining theorems 5 and 7.

Corollary 8. *Consider a spherically symmetric tree T with a degree sequence defined by (3). Then the SRW on T is*
(1) a.s. recurrent if $\overline{\lim}\, n\, q_n < \frac{1}{\log 2}$
(2) a.s. transient if $\underline{\lim}\, n\, q_n > \frac{1}{\log 2}$

In the following example, we present two sequences $\{q_n\}$ and $\{q'_n\}$ such that d_f in both cases equals 2 and we have recurrence for the first and transience for the second.

Example 1. *(1) Let $q_n = \frac{1}{n \log 2}$. It is known from [5] or [8] that the SRW on T is a.s. recurrent. Applying theorem 5, we obtain $d_f = 2$ a.s.*
(2) Let

$$q'_n = \frac{1}{n \log 2}\, [1 + \frac{3}{4} (\frac{1}{\log n})^{\frac{1}{4}} + o(\frac{1}{\log n})]$$

It was shown in [7] that the SRW on T with q'_n is a.s. transient. Obviously, $\lim_n n\, q'_n = \frac{1}{\log 2}$ and applying theorem 5 yields $d_f = 2$ a.s. △

4 Branching Processes in Varying Environments Trees.

The trees we consider in this section are such that the degrees of the vertices of the same generation are iid random variables and independent of the degrees of the vertices of the other generations. That is, if $d_{n,k}$ denotes the degree of the k^{th} vertex of the n^{th} generation, then we have a doubly indexed family $\{d_{n,k},\ n \geq 0,\ k \geq 1\}$ of independent random variables and for fixed n they are iid random variables. This is called Branching Processes in Varying Environments Trees and is abbreviated as BPVET. We use the symbol T^* for such trees and Z_n^* for the size of the n^{th} level.

The sequence $\{\frac{Z_n^*}{E(Z_n^*)}\}$ is a nonnegative martingale and, as such, converges a.s. to a finite limit W. Moreover, it follows from [9] that if the sequence $\{d_{n,k}\}$ is uniformly bounded, then there exists $W > 0$ a.s. such that

$$\frac{Z_n^*}{E(Z_n^*)} \longrightarrow W \quad a.s. \tag{7}$$

We consider a tree T^* with the degree sequence $\{d_{n,k}\}$ following the distribution (8). Let $d_{n,k}^+ = d_{n,k} - 1$ and

$$d_{n,k}^+ = \begin{cases} 1 & \text{with probability } 1 - q_n \\ \\ 2 & \text{with probability } q_n \end{cases} \tag{8}$$

where $0 < q_n < 1$. Obviously,

$$Z_{n+1}^* = \sum_{k=1}^{Z_n^*} d_{n,k}^+$$

and hence,

$$E(Z_n^*) = \prod_{k=0}^{n-1} (1 + q_k)$$

We notice that for spherically symmetric trees defined by (3) and BPVET defined by (8) have the same mean size, i.e. $E(Z_n) = E(Z_n^*) = M_n$. It is shown in [9] that the SRW on T^* is recurrent if and only if

$$\sum_n \frac{1}{M_n} = \infty \tag{9}$$

The type of the SRW on T^* for the case $q_n = \frac{c}{n}$ for some constant c was partially determined in [8] and completely settled, in a more general setting, in [9]. Here is a slightly generalized statement.

Theorem 9. *The SRW on a tree T^* with a degree sequence defined by (8) is a.s. recurrent if and only if* $\overline{\lim}\, n\, q_n \leq 1$.

Proof. We consider the case $q_n = \min(1, \frac{c}{n})$ for some positive constant c. Then,

$$M_n = \prod_{k=0}^{n-1} \min(1 + \frac{c}{k}, 2) \cong \alpha\, n^c$$

for some constant α. See [3, p.63]. Thus, from (9), the event of recurrence occurs a.s. if and only if $c \leq 1$. For the general case,

$$M_n = \prod_{k=0}^{n-1} (1 + q_k)$$

and the result follows from (9) and the comparison to $q_n = \frac{c}{n}$. ☐

The following theorem provides a formula of d_f for BPVET.

Theorem 10. *Consider a BPVET with a degree sequence defined by (8), such that $q_n \to 0$ as $n \to \infty$. Then,*

$$d_f = 1 + \lim_n n\, q_n \quad a.s.$$

Proof. It follows from (7) that

$$\begin{aligned} \lim_n \tfrac{\log Z_n^*}{\log n} &= \lim_n \tfrac{\log(\frac{Z_n^*}{E(Z_n)}) + \log E(Z_n)}{\log n} \\ &= \lim_n \tfrac{\log E(Z_n)}{\log n} = \lim_n \tfrac{\log \prod_{k=0}^{n-1}(1+q_k)}{\log n}. \end{aligned}$$

It follows from lemma 3 that

$$\lim_n \frac{\log Z_n^*}{\log n} = \lim_n n\, q_n \quad a.s.$$

Then lemma 4 assures that

$$d_f = 1 + \lim_n n\, q_n \quad a.s.$$

☐

The following corollary is an immediate consequence from theorems 9 and 10.

Corollary 11. *The SRW on T^* is a.s. recurrent if and only if $d_f \leq 2$ a.s.*

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