

Dimensions of Random Trees

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Abstract. In this paper we show, for Galton-Watson tree T of resistance R , that $R - R_n$ decays exponentially in n where R_n denotes the resistance of the portion of T between the root and level n . We also determine a formula for the resistance dimension of spherically symmetric random trees and prove that it is equal to the fractal dimension. We emphasize the relationship between these dimensions and the type, of being transient or recurrent, of the simple random walks on such trees.

1 Introduction.

The resistance dimension of a fixed graph has been studied by some authors. Telcs in his paper [13], obtained an interesting logarithmic form of the Einstein relation relating three exponent dimensions of smooth, polynomially growing graphs:

$$d_R = d_f + 2 - d_\Omega, \tag{1}$$

where d_R , d_f , d_Ω are random walk, fractal, and resistance dimensions, respectively. Zhou [15] presented some sort of generalization. In this paper, we pay our attention to study d_Ω for some types of random trees. All trees we consider are infinite and leafless (every vertex has at least two neighbors). We convert a tree T to an electric network by assigning a unit resistance to every edge of T . Let R_n be the resistance of the portion of T between the root r of T and the n^{th} level, S_n (vertices at distance n from the root). The resistance of the whole tree is defined to be, $R = \lim_n R_n$. This limit exists (possibly infinite) because

R_n is nondecreasing in n . The resistance dimension of T is defined as follows:

$$d_\Omega = \begin{cases} 2 - \limsup_{n \rightarrow \infty} \frac{\log R_n}{\log n} & \text{if } R_n \uparrow \infty \\ 2 - \limsup_{n \rightarrow \infty} \frac{\log(R - R_n)}{\log n} & \text{if } R_n \uparrow R < \infty \end{cases} \quad (2)$$

A tree is said to be spherically symmetric if all the vertices at the same distance from the root have the same degree. The resistance of a spherically symmetric tree is easy to calculate. It is known that the resistance R of such a tree is

$$R = \lim_n R_n = \lim_n \sum_{k=1}^n \frac{1}{Z_k} = \sum_{k=1}^{\infty} \frac{1}{Z_k} \quad (3)$$

where $Z_n = |S_n|$ is the number of vertices at distance n from the root. See [1] or [9].

A tree T is said to be homogeneous if its vertices are all of the same degree. Obviously, d_Ω for the homogeneous tree of degree 2 vertices equals 1, since $R_n = \frac{n}{2}$. While, it equals ∞ for that of degree 3 vertices, since $R_n = \frac{2}{3}(1 - \frac{1}{2^n})$.

In order to explore the phase transition between finite and infinite d_Ω , we chose to consider random trees in which the degrees of the vertices are i.i.d. random variables taking value 3 with probability q and 2 with probability $1 - q$. These are Galton-Watson trees. It turns out that all such trees have infinite d_Ω . Our claim is that trees for which the probability q is made to be level dependent and converges rapidly to zero might turn out to have finite d_Ω . In other words, if the degree 3 vertices become sparser as the tree grows up, we may obtain finite d_Ω . We consider trees for which every vertex of level n has degree 3 with probability q_n and degree 2 with probability $1 - q_n$. We determine, for such trees, the resistance dimension in terms of q_n . The fractal dimension d_f of a tree T is defined as,

$$d_f = \limsup_n \frac{\log b_n}{\log n}; \quad b_n = \sum_{k=1}^n Z_k. \quad (4)$$

We show that for any tree of polynomial growth $d_f \geq d_\Omega$, and we have equality for spherically symmetric trees. But we first consider the Galton-Watson trees.

Let us use the notation $a_n \asymp b_n$ if $\lim_n \frac{\log a_n}{\log b_n} = 1$. Generally, a graph is said to be polynomially growing if there exists a sequence $\{\alpha_n\}$ such that $\lim_n \alpha_n = \alpha > 0$ and $Z_n \asymp n^{\alpha_n}$.

2 Galton- Watson Trees.

We start with positive integer valued random variable X with $p(X = k) = p_k$, $k = 1, 2, 3, \dots$, and $m = \sum_k kp_k > 1$. We consider a supercritical branching process, starting from the first ancestor, and assume that parents produce randomly and independently offspring following the same law of X . Connect each parent to its offspring with edges. This defines rooted leafless infinite Galton-Watson trees and a measure $GW(dw)$ on such trees. The concept of flow on a graph is one of the recent tools that plays a vital role in deciding whether the effective resistance of an infinite electric network is finite and, as such, determining the type (of being transient or recurrent) of the simple random walk (SRW) on the underlying graph, since SRW on a graph is transient if and only if its effective resistance is finite. This will be introduced briefly at the end of section 3. See [1], [9], or [12]. A flow F on a finite graph G from a vertex a to a vertex b is defined to be an assignment of numbers F_{xy} to the edges xy of G such that (1) $F_{xy} = -F_{yx}$ (2) $\sum_y F_{xy} = 0$ if $x \neq a, b$ (3) $F_{xy} = 0$ if x and y are not neighbors. Let $d(r, y)$ denote the length of the unique path from r to y . We use the notation $x \rightarrow y$ if x is on the unique path from the root r to y such that $d(r, y) = d(r, x) + 1$. The flow out of a vertex x is defined to be:

$$F(x) = \sum_{y:x \rightarrow y} F_{xy} \quad (5)$$

If unit resistances are assigned to the edges of G and a voltage v_a is imposed between a and b such that the voltage v_b at b is 0, currents i_{xy} flow into the resistances. The energy dissipation in the circuit is

$$\frac{1}{2} \sum_{x,y} i_{xy}^2 \quad (6)$$

If the voltage v_a at a is adjusted so that one unit of the current flows into the circuit, the effective resistance, R_{eff} , of the circuit is defined to be v_a and it is shown to equal

$$R_{eff} = \frac{1}{2} \sum_{x,y} i_{xy}^2 \quad (7)$$

That is, the effective resistance of a circuit is the energy dissipation of the unit current flow. See [1, pp.61-62]. We should mention here that R and R_n in the introduction are effective resistances. **Thomson's Principle** assures that the effective resistance minimizes the energy dissipation $\frac{1}{2} \sum_{x,y} j_{xy}^2$ among all unit flows j from a to b . See [1, p.63]. The useful **cutting law** states that: Cutting certain branches off a graph can only increase

the effective resistance between two given nodes. Hereafter, $D(j)$ stands for the energy dissipation of the flow j . This discussion is useful in proving the following proposition.

Proposition 2.1. *Let T be an infinite tree such that the degree of every vertex is at least three. Then $R - R_n$ decays exponentially in n .*

We know that the homogeneous tree of degree 3 vertices has finite resistance and cutting principle guarantees finite resistance for every tree of the type described in proposition 2.1.

Proof. Let F_n be the unit current flow applied to the portion of T from the root r to the n^{th} level, S_n . It follows from Thomson's principle that $D(F_n) = R_n$. The subtrees of T hanging at every vertex has finite resistance and, as such, the current splits at every vertex. Whence, there is $0 < k < 1$ such that

$$\sup_{x \in S_n} F_n(x) \leq k^n \tag{8}$$

If the flow F_n is extended somehow on the whole tree to a flow F_n^* with finite energy, it follows then from Thomson's principle that,

$$\begin{aligned} R - R_n &\leq D(F_n^*) - D(F_n) \\ &= C \sum_{x \in S_n} F_n^2(x) \leq k^n C \sum_{x \in S_n} F_n(x) = Ck^n. \end{aligned}$$

for some constant C . The last equation follows since the current flowing into the circuit is the unity. This entails the proposition. \square

For the next theorem we need the following crucial technical lemma which is extracted from a lemma presented in [4] and reproduced in [14]. Let Γ denote the ensemble of all leafless rooted trees. Consider $T \in \Gamma$. A branching vertex of T is a vertex with at least three neighbors. let $N_n(x)$ denote the number of branching vertices on the unique path connecting the root r to a vertex $x \in S_n$.

Lemma 2.2. *Let T be a Galton-Watson tree with offspring mean m , $1 < m < \infty$. Then there is $\beta > 0$ such that*

$$\liminf_n \min_{x \in S_n} \frac{N_n(x)}{n} \geq \beta, \text{ GW - a.s.} \tag{9}$$

Theorem 2.3. *Let T be a Galton-Watson tree such that every vertex has degree 3 with probability q and degree 2 with probability $1 - q$. Then $R - R_n$ decays exponentially in n .*

Proof. Let F_n , as defined in proposition 2.1, be the unit current flow applied to the portion of T from r to S_n . Lemma 2.2 assures that, a.s. along every ray, a positive fraction greater than β of vertices have a subtree hanging off the ray with finite resistance. This allows a positive fraction of the current flows into each of such subtrees. Whence, there is $0 < k < 1$ and $0 < \alpha < \beta$ such that, for sufficiently large n ,

$$\sup_{x \in S_n} F_n(x) \leq k^{n\alpha} \quad a.s. \quad (10)$$

We extend the flow F_n to a flow F_n^* on the whole tree in the way that it splits equally at every branching vertex. It is known that this flow has a finite energy dissipation. See theorem 3.5. of [6]. Let \mathcal{F}_n be the σ -field generated by R_1, R_2, \dots, R_n and E_n denote the conditional expectation given \mathcal{F}_n . It follows from Thomson's principle that

$$\begin{aligned} E_n(R - R_n) &\leq E_n(D(F_n^*) - D(F_n)) \\ &\leq C_n E_n(\sum_{x \in S_n} F_n^2(x)) \\ &\leq C_n E_n(\sup_{x \in S_n} F_n(x) \sum_{x \in S_n} F_n(x)) \\ &= C_n E_n(\sup_{x \in S_n} F_n(x)) \leq C k^{n\alpha}. \end{aligned}$$

for some positive constants C_n and C . Let p_n denote the conditional probability given \mathcal{F}_n . Then the conditional Markov inequality [2, p.195] assures that

$$\sum_n p_n(R - R_n \geq \sqrt{k^{n\alpha}}) \leq \sum_n \frac{E_n(R - R_n)}{\sqrt{k^{n\alpha}}} < \infty$$

and the result follows from the conditional Borel-Cantelli lemma [3, p.325]. □

Note 1: The idea of this proof came out of a private communication with Yuval Peres.

Note 2: We conclude from this theorem that the resistance dimension of these trees is infinity and so it is not, for Galton -Watson trees, the appropriate notion to look at.

3 Resistance Dimension of Spherically Symmetric Random Trees.

This section is devoted to determining d_Ω for spherically symmetric random trees. We consider a sequence $\{d_n, n \geq 0\}$ of independent random variables where d_n refers to the

degree (the number of neighbors) of every vertex of level n . It is only worthwhile to consider a tree T where the distribution of d_n is

$$d_n = \begin{cases} 2 & \text{with probability } 1 - q_n \\ 3 & \text{with probability } q_n \end{cases} \quad (11)$$

Let $d_n^+ = d_n - 1$; $n \geq 1$ and set $d_0^+ = d_0$. If $Z_n = |S_n|$ denotes the number of vertices of level n , then

$$Z_n = \prod_{k=0}^{n-1} d_k^+ \quad (12)$$

To obtain d_Ω of these trees we first introduce the following lemmas.

Lemma 3.1. [9].

Suppose that X_n , $n = 1, 2, \dots$ are independent random variables with $0 \leq X_n \leq M$ for some constant M . Let

$$S_n = \sum_{k=1}^n X_k$$

If $E(S_n) \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\frac{S_n}{E(S_n)} \rightarrow 1 \quad \text{a.s.}$$

Lemma 3.2. [5, p.34].

Let $\{a_n\}$ and $\{b_n\}$ be two positive sequences such that $\sum_n b_n$ is divergent. If $\lim_n \frac{a_n}{b_n} = L$, then

$$\lim_n \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} = L \quad (13)$$

Lemma 3.3. If the positive sequence $\{a_n\}$ is nondecreasing, then

$$\lim_n na_n = \lim_n \frac{\sum_{k=1}^n a_k}{\log n} \quad (14)$$

Proof. It follows from lemma 3.2 that

$$\lim_n \frac{\sum_{k=1}^n a_k}{\log n} = \lim_n \frac{a_n}{1/n} = \lim_n na_n.$$

□

The following lemma is presented in [10] and we represent it here for the sake of completeness.

Lemma 3.4. *Let $\{a_n\}$ be a positive sequence such that $\lim_n \frac{\log a_n}{\log n} = L$ for some $L \geq 0$. Then*

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = L + 1 \quad (15)$$

Proof. If $L = \infty$, we have nothing to prove. So we may assume that $0 \leq L < \infty$. We first consider the case $L > 0$. It follows now that

$$a_n = n^{Lt_n}; \quad t_n \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

If $\epsilon > 0$ is arbitrary small, then for sufficiently large n , $L - \epsilon \leq Lt_n \leq L + \epsilon$ and hence for some constants A_1 and A_2 ,

$$A_1 + \sum_{k=1}^n k^{L-\epsilon} \leq \sum_{k=1}^n k^{Lt_k} \leq A_2 + \sum_{k=1}^n k^{L+\epsilon}$$

Consequently,

$$\lim_n \frac{\log \sum_{k=1}^n k^{L-\epsilon}}{\log n} \leq \lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = \lim_n \frac{\log \sum_{k=1}^n k^{Lt_k}}{\log n} \leq \lim_n \frac{\log \sum_{k=1}^n k^{L+\epsilon}}{\log n}$$

Choosing ϵ such that $L - \epsilon > 0$ and using the fact that for $\beta \geq 0$,

$$\lim_n \frac{\sum_{k=1}^n k^\beta}{n^{\beta+1}} = \frac{1}{\beta + 1} \quad (16)$$

we obtain,

$$L + 1 - \epsilon \leq \lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} \leq L + 1 + \epsilon$$

Whence,

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = L + 1$$

For the case $L = 0$, we obtain

$$1 - \epsilon \leq \lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} \leq 1 + \epsilon$$

where the right hand side inequality is obtained using the argument above and the left hand side one is obtained using the fact that for $0 < x \neq 1$,

$$\sum_{k=1}^n k^{-x} \geq \frac{(n+1)^{1-x} - 1}{1-x} \quad (17)$$

□

Note 3: The argument for the case $L = 0$ extends to the case $-1 \leq L \leq 0$. Whereas, if $L < -1$, the series $\sum_n a_n$ converges and the limit

$$\lim_n \frac{\log \sum_{k=1}^n a_k}{\log n} = 0 \quad (18)$$

Here is the result that determines the resistance dimension of spherically symmetric trees.

Theorem 3.5. *Let T be a spherically symmetric tree whose degree sequence $\{d_n\}$ is such that, for $0 < q_n < 1$,*

$$d_n = \begin{cases} 2 & \text{with probability } 1 - q_n \\ 3 & \text{with probability } q_n \end{cases} \quad (19)$$

Provided that $\lim_n n q_n$ exists, then

$$d_\Omega = 1 + (\log 2)(\lim_n n q_n) \quad (20)$$

Proof. Setting $d_n^+ = d_n - 1$, then

$$E(\log Z_n) = \sum_{k=0}^{n-1} E(\log d_k^+) = (\log 2) \sum_{k=0}^{n-1} q_k$$

Using lemmas 3.1 and 3.3, we obtain

$$\begin{aligned} \lim_n \frac{\log Z_n}{\log n} &= \lim_n \frac{\log Z_n}{E(\log Z_n)} \cdot \frac{E(\log Z_n)}{\log n} \\ &= \lim_n \frac{E(\log Z_n)}{\log n} \\ &= \lim_n \frac{(\log 2) \sum_{k=1}^n q_n}{\log n} = (\log 2) \lim_n n q_n. \end{aligned}$$

Whence,

$$\lim_n \frac{\log(1/Z_n)}{\log n} = -(\log 2) \lim_n n q_n$$

If $R_n \rightarrow \infty$, it follows from (24) below that $L = -\lim_n n q_n \log 2 \geq -1$. Whence, using (3) and lemma 3.4 (Note 3),

$$\lim_n \frac{\log R_n}{\log n} = 1 - (\log 2) \lim_n n q_n$$

Thus,

$$d_\Omega = 1 + (\log 2) \lim_n n q_n$$

If $R_n \rightarrow R < \infty$. Since

$$\lim_n \frac{\log Z_n}{\log n} = \log 2 \lim_n n q_n \quad a.s.$$

then let $Z_n = n^{t_n}$; $t_n = n q_n \log 2 \rightarrow L$ as $n \rightarrow \infty$. It is evident that $L \geq 1$, since $R < \infty$. If $\epsilon > 0$ is arbitrary small, then for sufficiently large n , $L - \epsilon \leq t_n \leq L + \epsilon$

First, if $L > 1$, then

$$\sum_{k=n}^{\infty} k^{-L-\epsilon} \leq \sum_{k=n}^{\infty} \frac{1}{Z_k} \leq \sum_{k=n}^{\infty} k^{-L+\epsilon}$$

and so

$$-L - \epsilon + 1 \leq \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} \leq -L + \epsilon + 1$$

which implies that

$$d_\Omega = 2 - \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} = 2 - 1 + L = 1 + \log 2 \lim_n n q_n$$

Second, if $L = 1$, then for arbitrary small $\epsilon > 0$ and sufficiently large n , $t_n \leq 1 + \epsilon$. Hence,

$$-\epsilon \leq \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} \leq 0$$

where the right inequality is due to the convergence of $\sum_n \frac{1}{Z_n}$. Consequently, $\lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} = 0$, and so

$$d_\Omega = 2 - \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} = 2 = 1 + \log 2 \lim_n n q_n$$

This completes the proof of the theorem. □

As it is mentioned in section 2, the type (of being transient or recurrent) of the SRW on infinite graphs is confined by the finiteness of the resistance of the corresponding electric networks. We outline this for infinite leafless trees as follows. A particle is moving on the vertices of a tree T making its jumps to one of its neighbors and each one of these neighbors has the same likelihood of being reached. That is, the SRW on T is a Markov chain with the set of vertices of T as the state space and the transition probabilities are such that

$$p(x, y) = \begin{cases} \frac{1}{d(x)} & \text{if } y \text{ is a neighbor of } x \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

where $d(x)$ is the degree of the vertex x which is the number of vertices incident with x . It follows from (3) that this SRW on a spherically symmetric T is transient if and only if

$$\sum_{n=1}^{\infty} \frac{1}{Z_n} < \infty \quad (22)$$

This is a special case of the general result of Nash-Williams [12]. See also [11]. It is proved in [9] and reproved in [7] that the SRW on a tree T defined by (19) is

$$\text{transient if } \liminf_n nq_n > \frac{1}{\log 2} \quad (23)$$

and

$$\text{recurrent if } \limsup_n nq_n < \frac{1}{\log 2} \quad (24)$$

This follows now as a direct consequence of theorem 3.5. For, if $\lim_n nq_n > \frac{1}{\log 2}$, then $d_\Omega > 2$ and $R_n \uparrow R$ and transience occurs a.s. Whereas, if $\lim_n nq_n < \frac{1}{\log 2}$, then $d_\Omega < 2$ and $R_n \uparrow \infty$ and recurrence occurs a.s. However, if $d_\Omega = 2$ we may have SRW of either type. The following two examples are considered in [8].

Example 1. If $q_n = \frac{1}{n \log 2}$, it is shown in [7] and [8] that the SRW on T is a.s. recurrent and theorem 3.5 gives $d_\Omega = 2$ a.s.

Example 2. If

$$q_n = \frac{1}{n \log 2} \left[1 + \frac{3}{4} \left(\frac{1}{\log n} \right)^{\frac{1}{4}} + o\left(\frac{1}{\log n} \right) \right] \quad (25)$$

it is also shown in [8] that the SRW on T is a.s. transient. Again theorem 3.5 gives that $d_\Omega = 2$ a.s.

4 Fractal Dimension of Spherically Symmetric Random Trees.

For sufficiently regular (smooth) graphs, the fractal dimension d_f along with the resistance dimension d_Ω is closely related to the random walk dimension d_R by the relation (1). We show in the following theorem that d_f of spherically symmetric trees of polynomial growth is equal to d_Ω . But first we introduce the following proposition for which we need to introduce the **Shorting principle**: shorting (connecting a given set of nodes together with perfectly conducting wires so that current can pass freely between them) certain sets of nodes together can only decrease the effective resistance of the network between two given nodes. See [1, p.100]. Obviously shorting principle plays its role in the opposite direction of the cutting principle. Shorting vertices of each level of any tree yields a graph with resistance

$$\sum_n \frac{1}{Z_n} \tag{26}$$

and shorting principle implies that the effective resistance R of any tree satisfies the inequality

$$R \geq \sum_n \frac{1}{Z_n} \tag{27}$$

Proposition 4.1. *For any infinite tree T of polynomial growth, $d_f \geq d_\Omega$.*

Note 4: This inequality holds true generally for smooth graphs. See [13].

Proof. Consider first the case $R_n \rightarrow \infty$. Then, applying (27) on the portion of T from r to level n , we obtain

$$\frac{\log b_n R_n}{\log n} \geq \frac{\log(\sum_{k=1}^n Z_k \sum_{k=1}^n \frac{1}{Z_k})}{\log n} \geq 2$$

The second inequality follows from Cauchy-Schwartz inequality. Therefore,

$$\frac{\log b_n}{\log n} \geq 2 - \frac{\log R_n}{\log n}$$

Secondly, suppose that $R_n \rightarrow R < \infty$. Then, Shorting principle implies that

$$\frac{\log[(\sum_{k=1}^n Z_k)(R - R_n)]}{\log n} \geq \frac{\log[(\sum_{k=1}^n Z_k)(\sum_{k=n+1}^{\infty} \frac{1}{Z_k})]}{\log n}$$

and so

$$\lim_n \frac{\log[(\sum_{k=1}^n Z_k)(R - R_n)]}{\log n} \geq \lim_n \frac{\log Z_n}{\log n} + 1 + \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n}, \text{ by lemma 3.4}$$

Using the assumption of the polynomial growth of T , there exists a sequence $\{\alpha_n\}$ such that $\alpha_n \rightarrow \alpha$ and $Z_n \asymp n^{\alpha_n}$. Since $R < \infty$, then $\alpha \geq 1$.

Now, if $\alpha > 1$, then an argument similar to that given in theorem 3.5 leads to

$$\lim_n \frac{\log Z_n}{\log n} + 1 + \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} = \alpha + 1 + 1 - \alpha = 2$$

and hence $d_f \geq d_{\Omega}$.

Also, if $\alpha = 1$, then

$$\lim_n \frac{\log \sum_{k=1}^n Z_k}{\log n} + \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{Z_k}}{\log n} = \lim_n \frac{\log Z_n}{\log n} + 1 + 0 = 2$$

and the result follows. □

Theorem 4.2. *If T is a spherically symmetric tree of polynomial growth, then $d_f = d_{\Omega}$.*

Proof. It follows from lemma 3.4 that

$$\lim_n \frac{\log Z_n}{\log n} = d_f - 1 \tag{28}$$

This is actually the converse of lemma 3.4 which holds true as long as the limit does exist. Assume now that $R_n \rightarrow \infty$, then

$$d_{\Omega} = 2 - \lim_n \frac{\log R_n}{\log n} = 2 - \lim_n \frac{\log \sum_{k=1}^n \frac{1}{Z_k}}{\log n} \tag{29}$$

It follows from (28) and by applying lemma 3.4 that

$$d_{\Omega} = 2 - (1 - d_f + 1) = d_f.$$

In the case that $R_n \rightarrow R < \infty$, and since T is polynomially growing tree, then there exists a sequence $\{\alpha_n\}$ such that $\alpha_n \rightarrow \alpha \geq 1$ and $Z_n \asymp n^{\alpha_n}$. Therefore,

$$\lim_n \frac{\log(R - R_n)}{\log n} = \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{k^{\alpha_k}}}{\log n}$$

If $\alpha > 1$, then for arbitrary small $\epsilon > 0$ and sufficiently large n , $\alpha_n \geq \alpha - \epsilon$. Therefore,

$$\lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{k^{\alpha_k}}}{\log n} \leq \lim_n \frac{\log \sum_{k=n}^{\infty} \frac{1}{k^{\alpha-\epsilon}}}{\log n} = -\alpha + \epsilon + 1$$

and hence

$$\lim_n \frac{\log(R - R_n)}{\log n} \leq -\alpha + 1$$

whence

$$d_{\Omega} \geq \alpha + 1$$

But, $d_f = \alpha + 1$, and so proposition 4.1 implies that $d_f = d_{\Omega}$.

If $\alpha = 1$, then $d_f = 2$ and $d_{\Omega} = 2$, which completes the proof. \square

The random walk dimension is defined to be

$$d_R = \limsup_n \frac{\log E_r(T_n)}{\log n} \tag{30}$$

where $E_r(T_n)$ is the mean number of steps that the SRW on T , starting from the root r , has to make to hit level n . It can easily be shown that spherically symmetric trees are smooth graphs. See [13] for the definition. Equation (1) leads us to conclude that $d_R = 2$ for all spherically symmetric trees of polynomial growth.

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