The Limiting Spectral Distribution of Random Block-Matrices

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Outline

1. Definitions and Preliminaries
2. Existence Theorem
3. Some Examples

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Definitions

Wigner Random Matrix

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

Where...

- $\mathbf{W}$ is $n \times n$ Hermitian matrix
- The entries w_{ij} are i. i. d. complex r. v. such that

$$E(w_{ij}) = 0 \text{ and } E(|w_{ij}|^2) = 1 \text{ for all } i < j$$

- The entries w_{ii} are i. i. d. real r. v. such that

$$E(w_{ii}) = 0 \text{ and } E(w_{ii}^2) = 1 \text{ for all } i$$

Definitions

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Definitions

Spectral Measure

- The spectral measure of an $n \times n$ Hermitian matrix $\mathbf{A}$ is

$$\mu_{\mathbf{A}} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

where

  - $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $\mathbf{A}$.
  - δ_x is the point mass at x.

- The limiting spectral measure of $\{\mathbf{A}_n\}$ is the weak limit of $\mu_{\mathbf{A}_n}$.

Example

Wigner Theorem

The almost sure limiting spectral measure of $\{\frac{1}{\sqrt{n}}\mathbf{W}_n\}$ is the semi-circle law

$$\gamma_1(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}(-2 \leq x \leq 2)dx$$

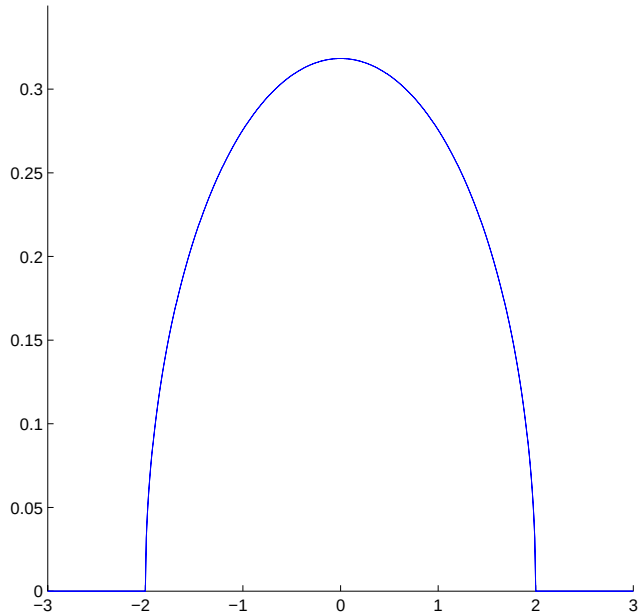

Figure: The semi-circle law.

Example

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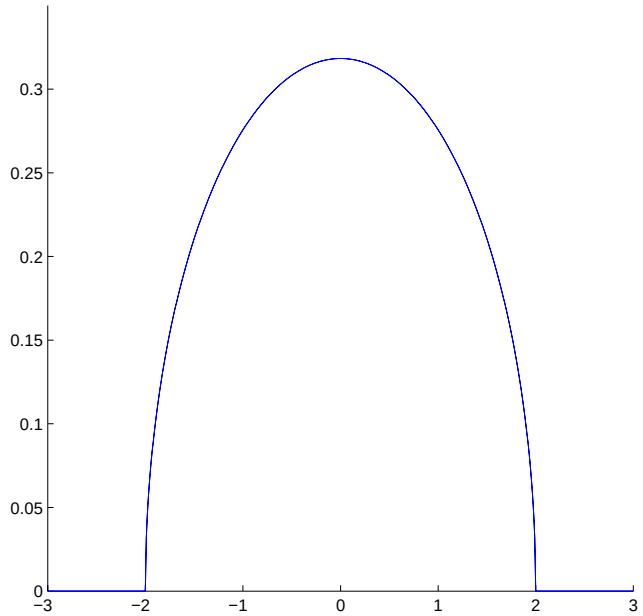

Figure: The semi-circle law.

$$\gamma_{\sigma^2}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}\ \mathbf{1}(-2\sigma \leq x \leq 2\sigma)dx$$

Definitions

Random Block-Matrices

A $k \times k$ Hermitian random block-matrix $\mathbb{B}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots)$ is a Hermitian matrix whose entries are the Hermitian random matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$.

If $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ are $n \times n$ Hermitian matrices, then $\mathbb{B}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots)$ is an $nk \times nk$ Hermitian matrix.

Example

Random Block-Matrices

The $k \times k$ Hermitian Toeplitz block-matrix defined as

$$\mathbb{T}^{(k)}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots & \mathbf{A}_{k-1} & \mathbf{A}_k \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_2 & \ddots & \ddots & \mathbf{A}_{k-1} \\ \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \ddots & \ddots & \cdot \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{A}_{k-1} & \ddots & \ddots & \ddots & \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_k & \mathbf{A}_{k-1} & \cdot & \cdots & \mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix}$$

Example

Random Block-Matrices

Consider Hermitian Toeplitz block-structures based on the three Hermitian matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$

$$\mathbb{L}^{(k)}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbb{T}^{(k)}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{B}, \mathbf{C}, \dots)$$

For instance,

$$\mathbb{L}^{(3)}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}, \ \mathbb{L}^{(4)}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}$$

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Existence Theorem

Theorem (Special Case-Oraby 2005)

Let $(\{\mathbf{A}_n(i)\}; i = 1, \ldots, h)$ *be a family of independent sequences of* $n \times n$ *Wigner random matrices. For a fixed* $k \times k$ *Hermitian block-structure* $\mathbb{B}$ *and each n, define*

$$\mathbf{X}_n := \mathbb{B}(\mathbf{A}_n(1), \mathbf{A}_n(2), \ldots, \mathbf{A}_n(h)).$$

Then there exists a non-random probability measure $\nu_{k,h,\mathbb{B}}$ *which depends only on* k, h *and the block structure* $\mathbb{B}$ *such that*

$$\mu_{\mathbf{X}_n} \xrightarrow{\omega} \nu_{k,h,\mathbb{B}} \quad \text{as } n \to \infty \qquad \text{a.s.}$$

Outline of the proof.

1. The proof is based on the method of moments.
2. Lemma. (Oraby 2005) For $s = 1, 2, \ldots$

$$\int_{\mathbb{R}} x^s \mu_{\mathbf{X}_n}(dx) \to \int_{\mathbb{R}} x^s \mu_{\mathbf{X}}(dx) \quad \text{as } n \to \infty \qquad a.s.$$

   where ...
   - $\mathbf{X} = \mathbb{B}(\mathbf{a}_1, \ldots, \mathbf{a}_h)$
   - $\mathbf{a}_1, \ldots, \mathbf{a}_h$ are free semicircle random variables in some noncommutative probability space.
3. For $s = 1, 2, \ldots$

$$\int_{\mathbb{R}} x^s \mu_{\mathbf{X}_n}(dx) = \mathrm{tr}_{kn}(\mathbf{X}_n^s)$$

4. $\mu_{\mathbf{X}}$ has a compact support in $\mathbb{R}$.

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Examples

$$\mathbb{L}^{(3)}(\mathbf{A},\mathbf{B},\mathbf{C}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}, \ \mathbb{L}^{(4)}(\mathbf{A},\mathbf{B},\mathbf{C}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}$$

The limiting spectral measure of $\mathbb{L}^{(2k)}$

Proposition.

Fix $k \geq 2$. If $\{\mathbf{A}_n\}, \{\mathbf{B}_n\}$ and $\{\mathbf{C}_n\}$ are three independent sequences of Wigner random matrices and for each n

$$\mathbb{L}_n^{(2k)} := \mathbb{L}^{(2k)}\left(\frac{1}{\sqrt{n}}\mathbf{A}_n, \frac{1}{\sqrt{n}}\mathbf{B}_n, \frac{1}{\sqrt{n}}\mathbf{C}_n\right)$$

then

$$\mu_{\mathbb{L}_n^{(2k)}} \xrightarrow{\omega} \frac{k-1}{k}\,\gamma_2 + \frac{1}{k}\,\gamma_{2(k^2-k+1)}$$

as $n \to \infty$, almost surely.

Examples

Case $k = 2$

$$\mu_{\mathbb{L}_n^{(4)}} \xrightarrow{\omega} \frac{1}{2}\,\gamma_2 + \frac{1}{2}\,\gamma_6 \quad \text{as } n \to \infty \qquad a.s.$$

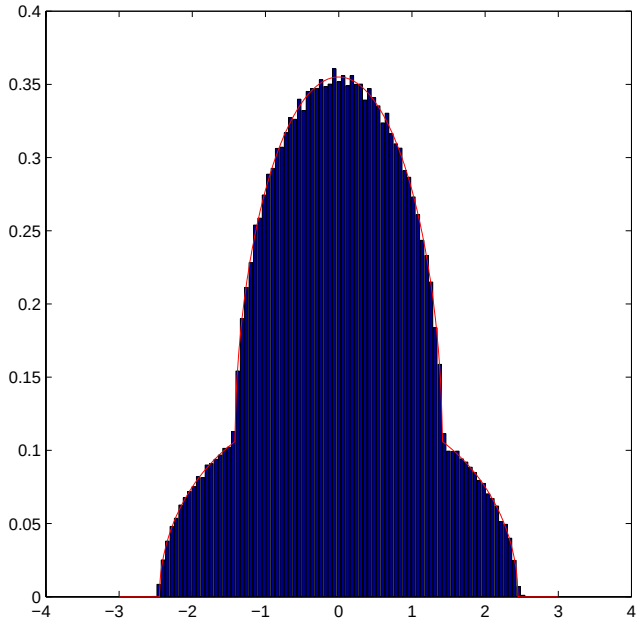


Figure: The limiting distribution of $\mu_{\mathbb{L}_n^{(4)}}$.

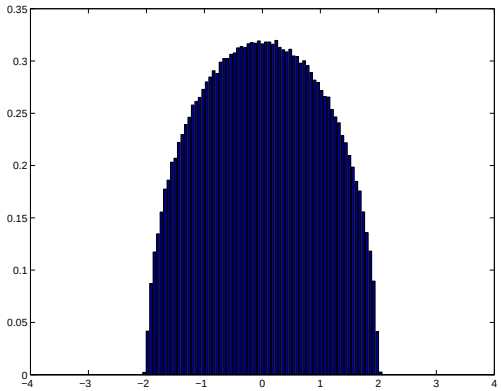
Simulations

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}$$

$$\mathbb{L}^{(2)} \qquad \mathbb{L}^{(3)} \qquad \mathbb{L}^{(4)}$$

Figure: Empirical histograms of 100 $\mathbb{L}^{(m)}$ matrices of $n = 200$.

Simulations



$\mathbb{L}^{(2)}$

$$\begin{bmatrix} A & B & C \\ B & A & B \\ C & B & A \end{bmatrix}$$

$\mathbb{L}^{(3)}$

$$\begin{bmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{bmatrix}$$

$\mathbb{L}^{(4)}$

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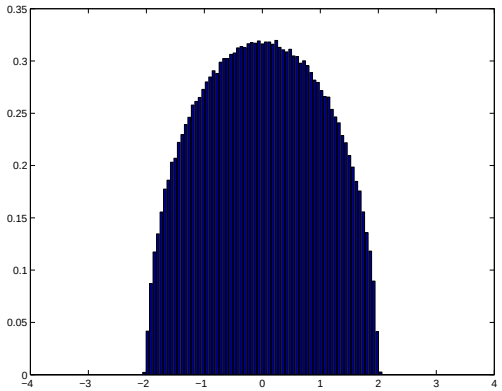

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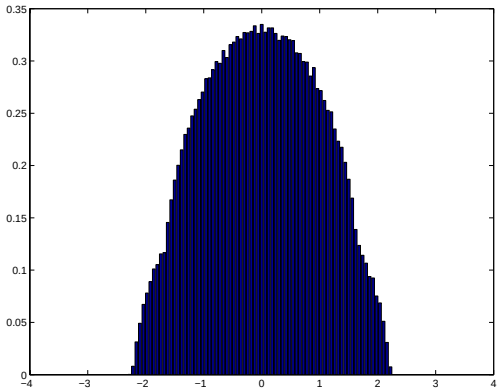

$\mathbb{L}^{(3)}$

$$\begin{bmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{bmatrix}$$

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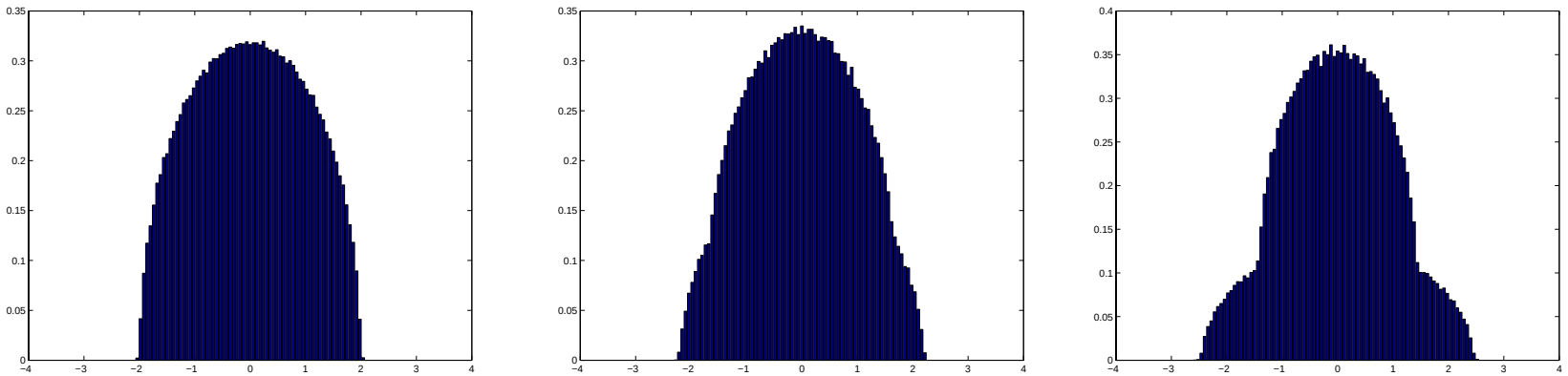

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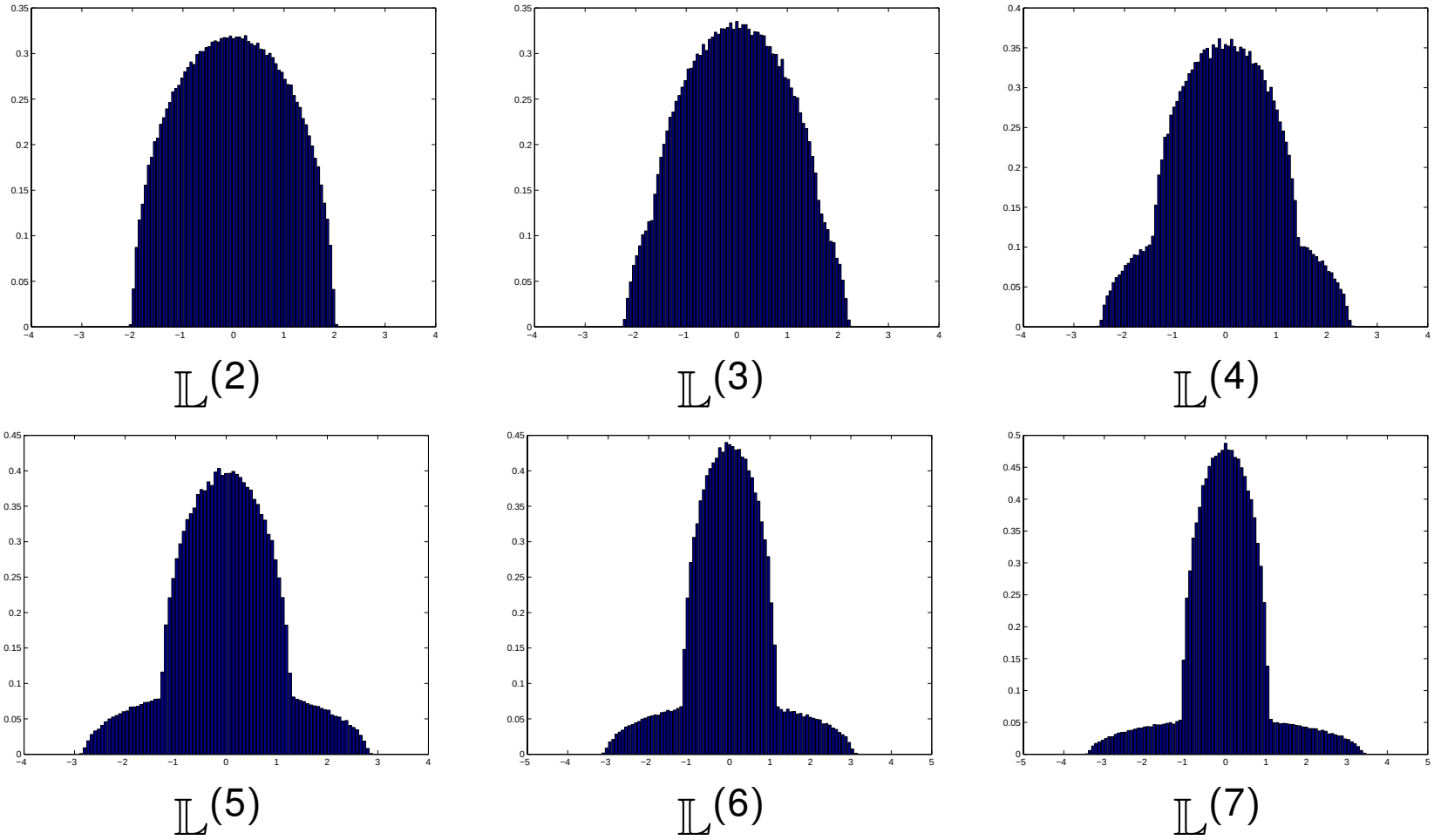

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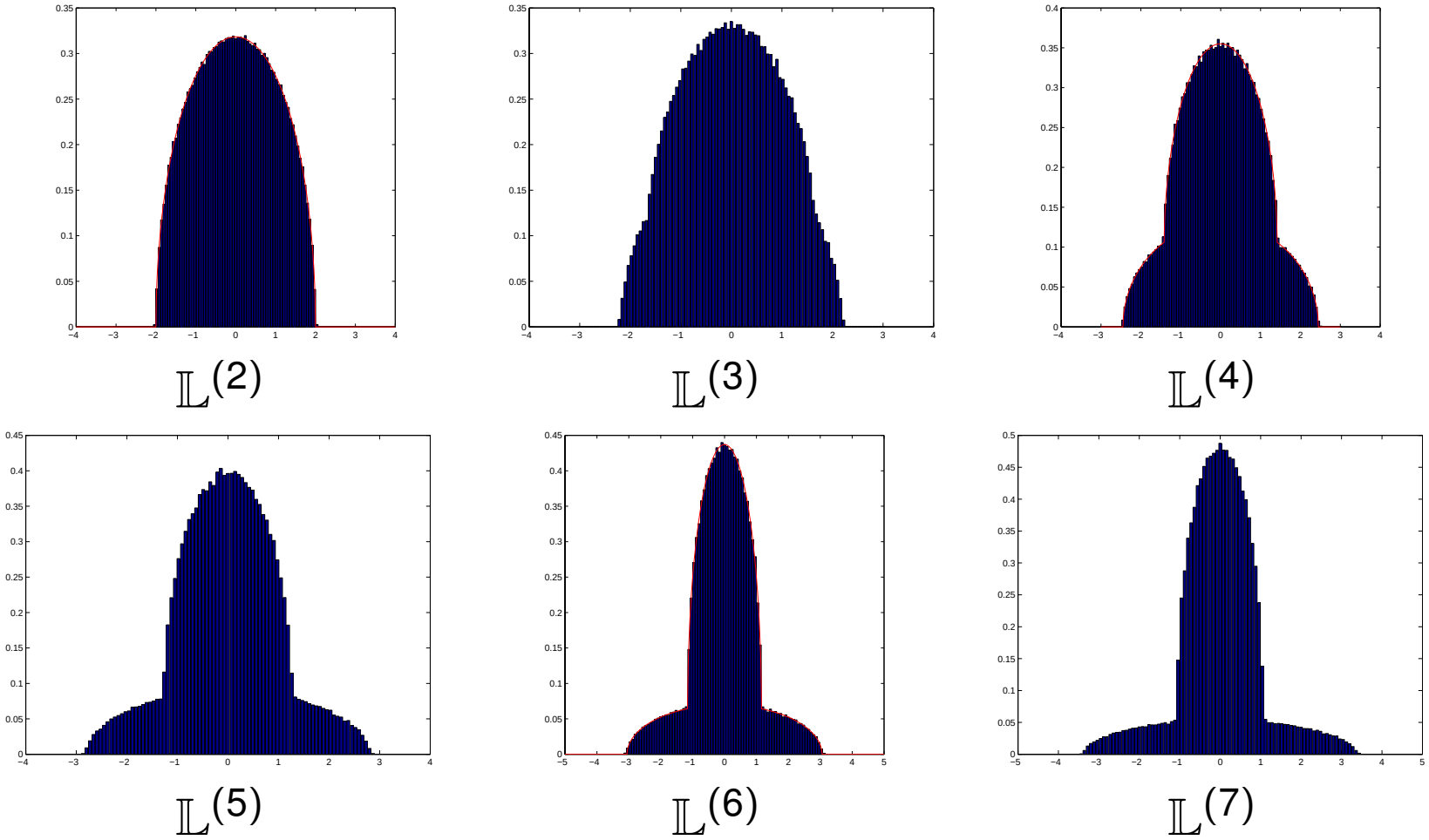


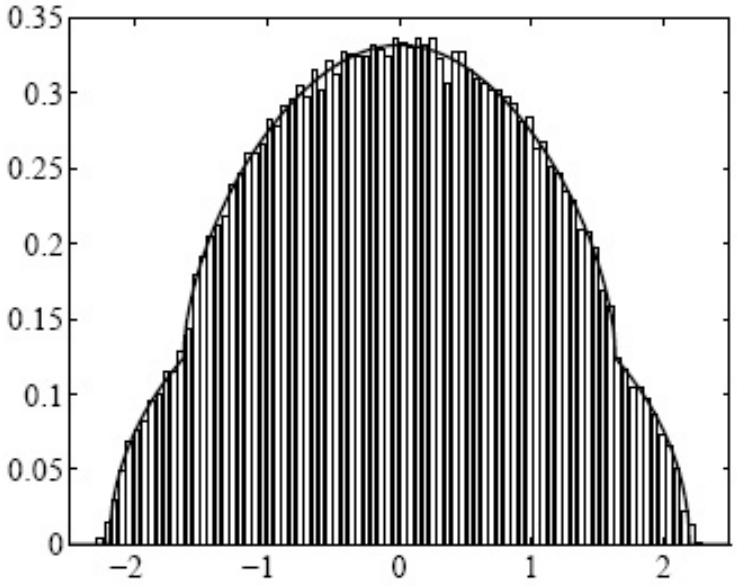
Figure: Empirical histograms of 100 $\mathbb{L}^{(m)}$ matrices of $n = 200$.

$\mathbb{L}^{(3)}$ and others

Far, R., Oraby, T., Bryc, W. and Speicher, R.
Spectra of Large Block Matrices.
Submitted, 2006.

- Operator-valued free probability.
- The Cauchy transform of a measure μ

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \, \mu(dx)$$

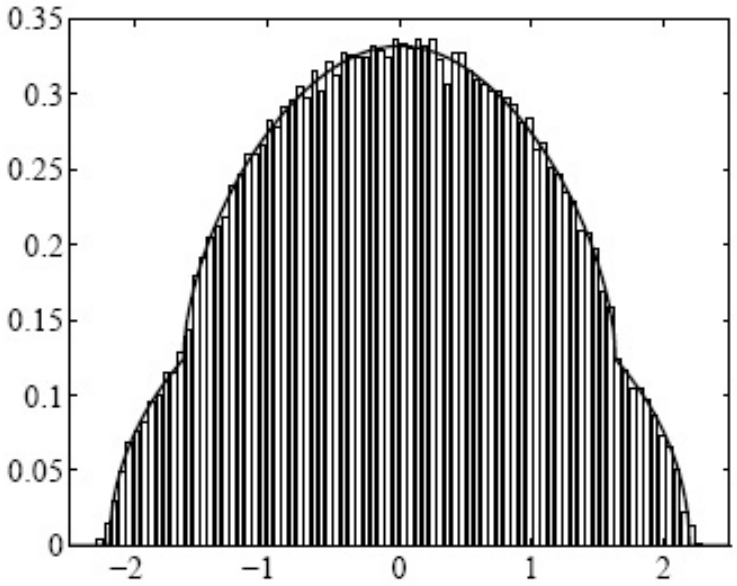


$\mathbb{L}^{(3)}$ and others

Far, R., Oraby, T., Bryc, W. and Speicher, R.
Spectra of Large Block Matrices.
Submitted, 2006.

- $G_\mu(z) = \frac{2}{3} f(z) + \frac{1}{3} g(z)$
- System of equations:

$$\begin{aligned} z\,f &= 1 + \frac{g\,(f+h) + 2\,(f^2+h^2)}{3} \\ z\,g &= 1 + \frac{g\,(g+2\,(f+h))}{3} \\ z\,h &= \frac{4\,f\,h + g\,(f+h)}{3} \end{aligned}$$



$\mathbb{L}^{(3)}$ and others

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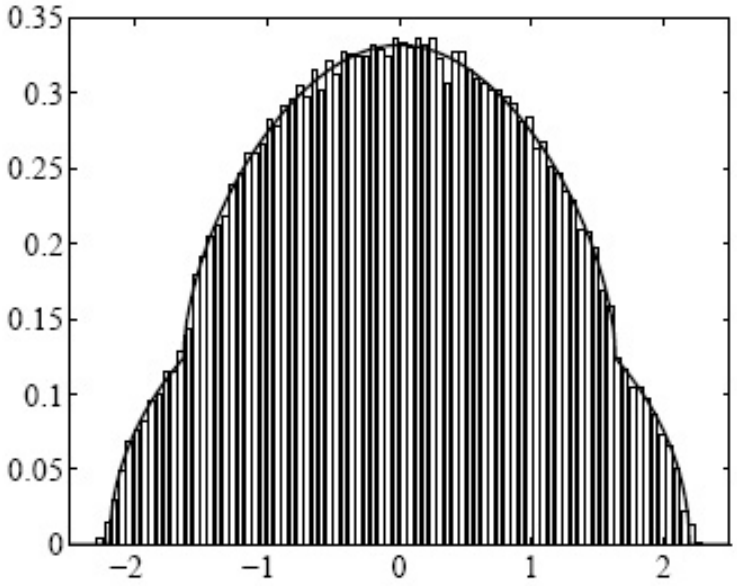
Spectra of Large Block Matrices.
Submitted, 2006.

- Stieltjes inversion formula

$$\mu(dx) = -\frac{1}{\pi} \lim_{y \downarrow 0} \Im G_\mu(x + \mathbf{i}y)dx$$



Examples

Bai (1999)

"*It could be that Theorems ... are still true when the underlying variables defining the Wigner or ... are weakly dependent, say ϕ-mixing, ...*"

- Schenker and Schulz-Baldes (2005)
- Oraby (2005)
- Anderson and Zeitouni (2006)

Examples

Example (Oraby 2005)

$$\mathbb{L}_n^{(4)} = \left[\begin{array}{c|c|c|c} \mathbf{A}_n & \mathbf{B}_n & \mathbf{C}_n & \mathbf{B}_n \\ \hline \mathbf{B}_n & \mathbf{A}_n & \mathbf{B}_n & \mathbf{C}_n \\ \hline \mathbf{C}_n & \mathbf{B}_n & \mathbf{A}_n & \mathbf{B}_n \\ \hline \mathbf{B}_n & \mathbf{C}_n & \mathbf{B}_n & \mathbf{A}_n \end{array}\right]$$

$$\mu_{\mathbb{L}_n^{(4)}} \xrightarrow{\omega} \frac{1}{2}\,\gamma_2 + \frac{1}{2}\,\gamma_6 \quad \text{as } n \to \infty \qquad a.s.$$

Examples

Example (Cont'd)

For $n = 3$

$$\mathbb{L}^{(4)} = \left[\begin{array}{ccc|ccc|ccc|ccc}
a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13} & c_{11} & c_{12} & c_{13} & b_{11} & b_{12} & b_{13} \\
a_{12} & a_{22} & a_{23} & b_{12} & b_{22} & b_{23} & c_{12} & c_{22} & c_{23} & b_{12} & b_{22} & b_{23} \\
a_{13} & a_{23} & a_{33} & b_{13} & b_{23} & b_{33} & c_{13} & c_{23} & c_{33} & b_{13} & b_{23} & b_{33} \\
\hline
b_{11} & b_{12} & b_{13} & a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13} & c_{11} & c_{12} & c_{13} \\
b_{12} & b_{22} & b_{23} & a_{12} & a_{22} & a_{23} & b_{12} & b_{22} & b_{23} & c_{12} & c_{22} & c_{23} \\
b_{13} & b_{23} & b_{33} & a_{13} & a_{23} & a_{33} & b_{13} & b_{23} & b_{33} & c_{13} & c_{23} & c_{33} \\
\hline
c_{11} & c_{12} & c_{13} & b_{11} & b_{12} & b_{13} & a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13} \\
c_{12} & c_{22} & c_{23} & b_{12} & b_{22} & b_{23} & a_{12} & a_{22} & a_{23} & b_{12} & b_{22} & b_{23} \\
c_{13} & c_{23} & c_{33} & b_{13} & b_{23} & b_{33} & a_{13} & a_{23} & a_{33} & b_{13} & b_{23} & b_{33} \\
\hline
b_{11} & b_{12} & b_{13} & c_{11} & c_{12} & c_{13} & b_{11} & b_{12} & b_{13} & a_{11} & a_{12} & a_{13} \\
b_{12} & b_{22} & b_{23} & c_{12} & c_{22} & c_{23} & b_{12} & b_{22} & b_{23} & a_{12} & a_{22} & a_{23} \\
b_{13} & b_{23} & b_{33} & c_{13} & c_{23} & c_{33} & b_{13} & b_{23} & b_{33} & a_{13} & a_{23} & a_{33}
\end{array}\right]$$

$$\mu_{\mathbb{L}_n^{(4)}} \xrightarrow{\omega} \frac{1}{2}\,\gamma_2 + \frac{1}{2}\,\gamma_6 \quad \text{as } n \to \infty \qquad a.s.$$

Examples

Example (Cont'd)

$$\mathbf{X} = \left[\begin{array}{cccc|cccc|cccc}
a_{11} & b_{11} & c_{11} & b_{11} & a_{12} & b_{12} & c_{12} & b_{12} & a_{13} & b_{13} & c_{13} & b_{13} \\
b_{11} & a_{11} & b_{11} & c_{11} & b_{12} & a_{12} & b_{12} & c_{12} & b_{13} & a_{13} & b_{13} & c_{13} \\
c_{11} & b_{11} & a_{11} & b_{11} & c_{12} & b_{12} & a_{12} & b_{12} & c_{13} & b_{13} & a_{13} & b_{13} \\
b_{11} & c_{11} & b_{11} & a_{11} & b_{12} & c_{12} & b_{12} & a_{12} & b_{13} & c_{13} & b_{13} & a_{13} \\
\hline
a_{12} & b_{12} & c_{12} & b_{12} & a_{22} & b_{22} & c_{22} & b_{22} & a_{23} & b_{23} & c_{23} & b_{23} \\
b_{12} & a_{12} & b_{12} & c_{12} & b_{22} & a_{22} & b_{22} & c_{22} & b_{23} & a_{23} & b_{23} & c_{23} \\
c_{12} & b_{12} & a_{12} & b_{12} & c_{22} & b_{22} & a_{22} & b_{22} & c_{23} & b_{23} & a_{23} & b_{23} \\
b_{12} & c_{12} & b_{12} & a_{12} & b_{22} & c_{22} & b_{22} & a_{22} & b_{23} & c_{23} & b_{23} & a_{23} \\
\hline
a_{13} & b_{13} & c_{13} & b_{13} & a_{23} & b_{23} & c_{23} & b_{23} & a_{33} & b_{33} & c_{33} & b_{33} \\
b_{13} & a_{13} & b_{13} & c_{13} & b_{23} & a_{23} & b_{23} & c_{23} & b_{33} & a_{33} & b_{33} & c_{33} \\
c_{13} & b_{13} & a_{13} & b_{13} & c_{23} & b_{23} & a_{23} & b_{23} & c_{33} & b_{33} & a_{33} & b_{33} \\
b_{13} & c_{13} & b_{13} & a_{13} & b_{23} & c_{23} & b_{23} & a_{23} & b_{33} & c_{33} & b_{33} & a_{33}
\end{array}\right]$$

$\mathbf{X}$ is similar to $\mathbb{L}^{(4)}$

Examples

Example (Cont'd)

For each n

$$\mathbf{x}_n = \left[\begin{array}{cccc|cccccc|cccc}
a_{11} & b_{11} & c_{11} & b_{11} & & & & & & & a_{1n} & b_{1n} & c_{1n} & b_{1n} \\
b_{11} & a_{11} & b_{11} & c_{11} & & \cdot & \cdot & \cdot & \cdot & & b_{1n} & a_{1n} & b_{1n} & c_{1n} \\
c_{11} & b_{11} & a_{11} & b_{11} & & & & & & & c_{1n} & b_{1n} & a_{1n} & b_{1n} \\
b_{11} & c_{11} & b_{11} & a_{11} & & & & & & & b_{1n} & c_{1n} & b_{1n} & a_{1n} \\
\hline
 & \vdots & & & & \cdot & & & & & & & \vdots & \\
 & & & & & & \cdot & & & & & & & \\
 & & & & & & & \cdot & & & & & & \\
 & & & & & & & & \cdot & & & & & \\
\hline
a_{1n} & b_{1n} & c_{1n} & b_{1n} & & & & & & & a_{nn} & b_{nn} & c_{nn} & b_{nn} \\
b_{1n} & a_{1n} & b_{1n} & c_{1n} & & & & & & & b_{nn} & a_{nn} & b_{nn} & c_{nn} \\
c_{1n} & b_{1n} & a_{1n} & b_{1n} & & \cdot & \cdot & \cdot & \cdot & & c_{nn} & b_{nn} & a_{nn} & b_{nn} \\
b_{1n} & c_{1n} & b_{1n} & a_{1n} & & & & & & & b_{nn} & c_{nn} & b_{nn} & a_{nn}
\end{array}\right]$$

$$\mu\mathbf{x}_n \xrightarrow{\omega} \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_6 \quad \text{as } n \to \infty \qquad a.s.$$

Thank you