Spectra of Random Block-Matrices

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Outline

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Definitions
Wigner Random Matrix

$$\mathbf{W} = \frac{1}{\sqrt{n}} \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

Where...

- $\mathbf{W}$ is $n \times n$ Hermitian matrix
- w_{ij}; $i \leq j$ are i. i. d. for all $1 \leq i < j \leq n$, with mean 0 and variance 1

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Wishart Random Matrix

$$\mathbf{X} = \frac{1}{n} A^* A$$

Where...

- $\mathbf{A}$ is a $p \times n$ matrix
- A_{ij} are independent Gaussian random variables such that $\Re A_{ij}, \Im A_{ij} \sim N(0, \frac{1}{2})$ for every i, j

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Spectral Measure

- The spectral measure of an $n \times n$ Hermitian matrix $\mathbf{A}$ is

$$\mu_{\mathbf{A}} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

where

- $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $\mathbf{A}$.
- δ_x is the point mass at x.

- $L \subset \mathbb{R}$,

$$\frac{1}{n} \sharp\{\text{e. v. in } L\} = \int_L \mu_{\mathbf{A}}(dx)$$

Spectral Measure (Cont'd)

- For every $k \geq 1$

$$\int_{\mathbb{R}} x^k \mu_{\mathbf{A}}(dx) = \frac{1}{n} Trace(\mathbf{A}^k)$$

- The limiting spectral measure of $\{\mathbf{A}_n\}$ is the weak limit of $\mu_{\mathbf{A}_n}$.

Simulations

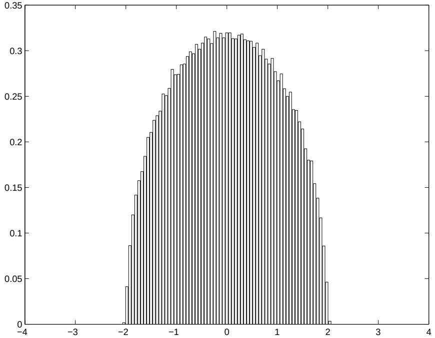

Figure: Wigner matrix repeated 100 times with $n = 300$

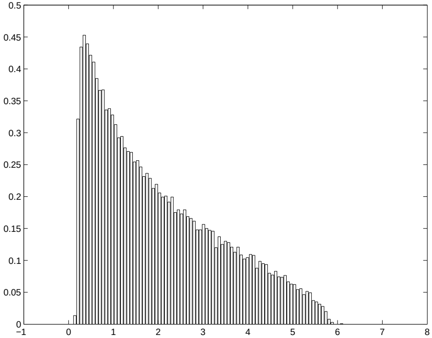

Figure: Wishart matrix repeated 100 times with $n = 300$ and $p = 2n$

Wigner's Law. Wigner (1955, 1958), Bai (1999)

The almost sure limiting spectral measure of a sequence of Wigner matrices $\{\mathbf{W}_n\}$ is the semi-circle distribution

$$\gamma_1(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\;\mathbf{1}_{[-2\leq x\leq 2]}(x)dx$$

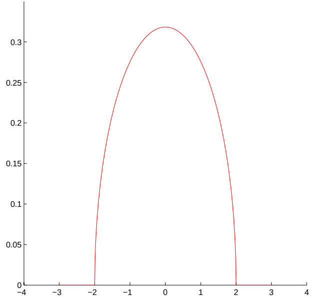

Figure: The semi-circle distribution.

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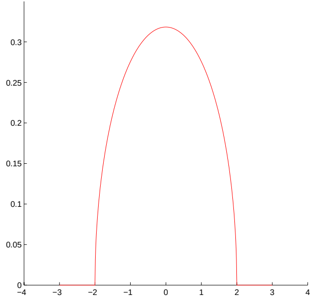

Figure: The semi-circle distribution.

$$\gamma_{\sigma^2}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\ \mathbf{1}_{[-2\sigma \le x \le 2\sigma]}(x)dx$$

Marchenko-Pastur's Law. Marchenko and Pastur(1967), Bai (1999)

The almost sure limiting spectral measure of a sequence of Wishart matrices $\{\mathbf{X}_n\}$ such that $\lim_{n\to\infty} \frac{p_n}{n} = \alpha > 0$ is the Marchenko-Pastur distribution with mean α

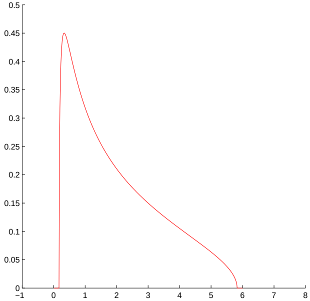

Figure: The Marchenko-Pastur distribution with $\alpha = 2$.

$$\rho_\alpha(dx) = \max(1-\alpha, 0)\delta_0(dx) + \frac{\sqrt{(x-(\sqrt{\alpha}-1)^2)((\sqrt{\alpha}+1)^2 - x)}}{2\pi x} \mathbf{1}_{[(\sqrt{\alpha}-1)^2, (\sqrt{\alpha}+1)^2]}(x)dx.$$

Free Probability Theory

We say that a family of sequences of random matrices $(\{\mathbf{A}_n(l)\}; l = 1, \dots, h)$ is asymptotically free if for every noncommutative polynomial p in h variables

$$\mathrm{tr}_n\left(p(\mathbf{A}_n(1), \dots, \mathbf{A}_n(h))\right) \xrightarrow{n\to\infty} \tau\left(p(\mathbf{a}_1, \dots, \mathbf{a}_h)\right) \qquad a.s.$$

where $(\mathbf{a}_1, \dots, \mathbf{a}_h)$ is a family of free noncommutative random variables in some noncommutative probability space $(\mathcal{A}, \tau)$

Free Probability Theory

Free additive convolution

- If $\{\mathbf{A}_n\}$ and $\{\mathbf{B}_n\}$ are asymptotically free
- $\mu_{\mathbf{A}_n} \xrightarrow{\mathfrak{m}} \mu$ as $n \to \infty$ *a.s.*

  $\mu_{\mathbf{B}_n} \xrightarrow{\mathfrak{m}} \nu$ as $n \to \infty$ *a.s.*

  where μ and ν are compactly supported in $\mathbb{R}$
- then

$$\mu_{\mathbf{A}_n + \mathbf{B}_n} \xrightarrow{\omega} \mu \boxplus \nu \quad \text{as } n \to \infty \qquad a.s.$$

Free Probability Theory

Example

- $\{\mathbf{A}_n\}$ Wigner matrices

$$\mu_{\mathbf{A}_n} \xrightarrow{\omega} \gamma_1 \quad \text{as } n \to \infty \qquad a.s.$$

- $\{\mathbf{B}_n\}$ Wishart matrices

$$\mu_{\mathbf{B}_n} \xrightarrow{\omega} \rho_1 \quad \text{as } n \to \infty \qquad a.s.$$

if $\lim_{n\to\infty} \frac{p_n}{n} = 1$.

- If $\mathbf{A}_n$ and $\mathbf{B}_n$ are independent, then

$$\mu_{\mathbf{A}_n+\mathbf{B}_n} \xrightarrow{\omega} \gamma_1 \boxplus \rho_1 \quad \text{as } n \to \infty \qquad a.s.$$

Free Probability Theory

Example (Cont'd)

- T.O. (2006)

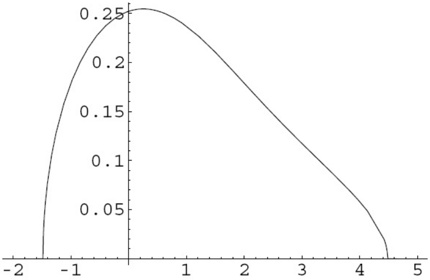

Figure: The probability density functions corresponding to $\gamma_1 \boxplus \rho_1$.

Free Probability Theory

Example (Cont'd)

For $t \neq 0$, $(\gamma_1 \boxplus D_t(\rho_1))(dx) = f(x;t)dx$

$$f(x;t) = \frac{1}{2\sqrt{3}\,\pi\,t}\left(H(x;t) - \frac{h_2(x;t)}{H(x;t)}\right) \mathbf{1}_{[s_1(t),s_2(t)]}(x),$$

$$h_1(x;t) = 2 + 27\,t^2 - 3\,t\,x - 3\,t^2\,x^2 + 2\,t^3\,x^3,$$

$$h_2(x;t) = 1 - t\,x + t^2\,x^2,$$

$$H(x;t) = \frac{1}{\sqrt[3]{2}}\left(h_1(x;t) + \sqrt{h_1(x;t)^2 - 4\,h_2(x;t)^3}\right)^{\frac{1}{3}},$$

Free Probability Theory

Example (Cont'd)

$s_1(t)$ and $s_2(t)$ are the two real roots of the quartic equation in x

$$4 + 27\,t^2 - 6\,t\,x - x^2 - 6\,t^2\,x^2 + 2\,t\,x^3 + 4\,t^3\,x^3 - t^2\,x^4 = 0$$

Random Block-Matrices

A $k \times k$ Hermitian random block-matrix $\mathbb{B}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots)$

is a Hermitian matrix whose entries are

the random matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$.

Random Block-Matrices
Examples

A, **B** and **C** are $n \times n$ matrices ...

$$\mathbb{T}^{(3)}(\mathbf{A},\mathbf{B},\mathbf{C}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}, \quad \mathbb{C}^{(4)}(\mathbf{A},\mathbf{B},\mathbf{C}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{A} \end{bmatrix}$$

Outline

Girko's Random Block-Matrix. Girko (2000)

$$X_{n,k} = \begin{bmatrix} A + E_{11} & E_{12} & \dots & E_{1k} \\ E_{12} & A + E_{22} & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1k} & E_{2k} & \dots & A + E_{kk} \end{bmatrix}$$

Where...

- The matrix A is an $n \times n$ non-random symmetric matrix
- The blocks E_{ij} are i. i. d. random matrices

$$P(E_{ij} = B) = P(E_{ij} = -B) = \frac{1}{2}$$

- The matrix B is an $n \times n$ non-random positive definite symmetric matrix

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Generalization of Girko's Block-Matrix. T.O. (2006)

$$X_{n,k} = \begin{bmatrix} A + w_{11}B & w_{12}B & \dots & w_{1k}B \\ w_{21}B & A + w_{22}B & \dots & w_{2k}B \\ \vdots & \vdots & \ddots & \vdots \\ w_{k1}B & w_{k2}B & \dots & A + w_{kk}B \end{bmatrix}$$

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Where...

- The matrix A is an $n \times n$ random Hermitian matrix
- The matrix B is an $n \times n$ random Hermitian matrix
- The entries w_{ij} are i. i. d. random variables. Assume $\mathbf{W}_k := (w_{ij})_{1 \le i,j \le k}$ is Hermitian

Simulations (1/2)

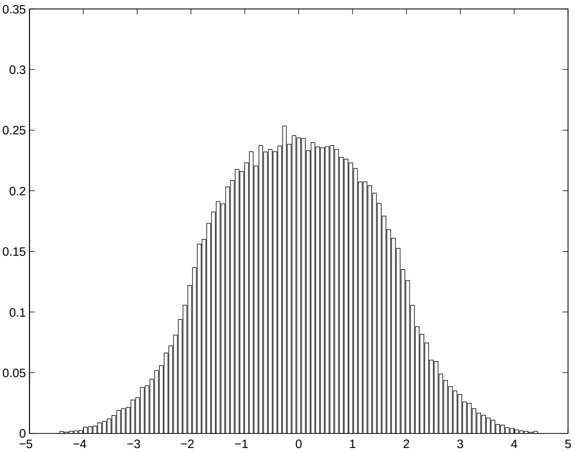

Figure: Histogram of the eigenvalues of $X_{40,40}$ when B_{40} is a Gaussian Wigner matrix.

Simulations (2/2)

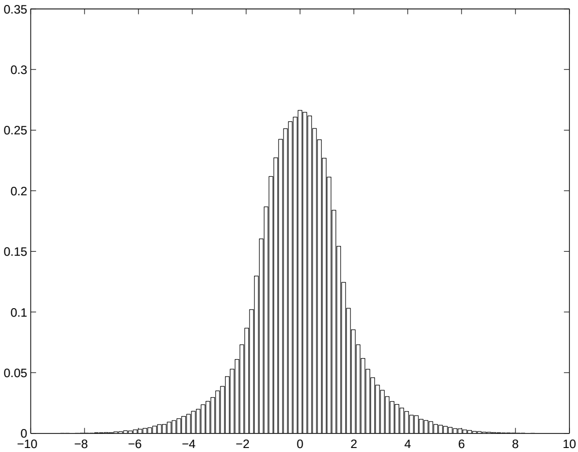

Figure: Histogram of the eigenvalues of $X_{40,40}$ when B_{40} is a Wishart matrix with $\frac{p}{n} = 1$.

The Limiting Spectral Measure

Theorem. T.O. (2006) Assumptions

1. There exists a compactly supported probability measure μ such that
$$\mu_{\mathbf{W}_n} \xrightarrow{\mathfrak{m}} \mu \text{ as } n \to \infty \qquad a.s.$$
2. For real t, there exist probability measures $\mathbf{K}(t;.)$ such that
$$\mu_{\mathbf{A}_n + t\mathbf{B}_n} \xrightarrow{\mathfrak{m}} \mathbf{K}(t;.) \text{ as } n \to \infty \qquad a.s.$$
and $\mathbf{K}(t;.)$ has a support that is uniformly bounded for t in any compact subset of $\mathbb{R}$.

The Limiting Spectral Measure (Cont'd)

Under these conditions we have

$$\lim_{n\to\infty}\lim_{k\to\infty}\mu_{\mathbf{X}_{n,k}} = \lim_{k\to\infty}\lim_{n\to\infty}\mu_{\mathbf{X}_{n,k}} \overset{\omega}{=} \nu \qquad a.s.$$

where the probability measure ν is defined as

$$\nu(dx) = \int_{\mathbb{R}} \mathbf{K}(t; dx)\,\mu(dt).$$

Corollary (1/2)

- $\{\mathbf{W}_n\}$ Wigner matrices
- $\{\mathbf{A}_n\}$ and $\{\mathbf{B}_n\}$ independent Wigner matrices

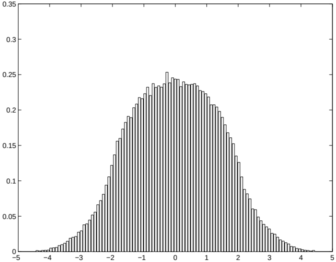

Figure: Histogram of the eigenvalues of $X_{40,40}$ when A_{40} and B_{40} are Gaussian matrices.

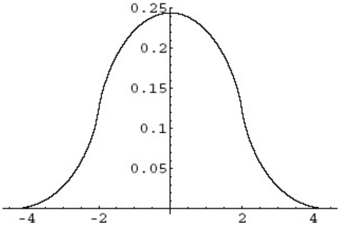

Figure: The probability density function corresponding to the limiting spectral distribution of $X_{n,k}$ when A and B are Wigner matrices.

Corollary (2/2)

- $\{\mathbf{W}_n\}$ Wigner matrices
- $\{\mathbf{A}_n\}$ and $\{\mathbf{B}_n\}$ independent & $\mathbf{A}_n$ is Wigner and $\mathbf{B}_n$ is Wishart
- $$g(x) := \int_{\mathbb{R}} f(x;t)\,\gamma_1(dt)$$
  where $f(x;t)$ is the pdf corresponding to $\gamma_1 \boxplus D_t(\rho_1)$.

Outline

Circulant Random Block-Matrices

The $k \times k$ Hermitian Circulant block-matrix defined as

$$\mathbb{C}^{(k)}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k) = \frac{1}{\sqrt{k}} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \ldots & \mathbf{A}_{k-1} & \mathbf{A}_k \\ \mathbf{A}_k & \mathbf{A}_1 & \mathbf{A}_2 & \ddots & \ddots & \mathbf{A}_{k-1} \\ \mathbf{A}_{k-1} & \mathbf{A}_k & \mathbf{A}_1 & \ddots & \ddots & . \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{A}_3 & \ddots & \ddots & \ddots & \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_3 & . & \ldots & \mathbf{A}_k & \mathbf{A}_1 \end{bmatrix}$$

such that $\mathbf{A}_i = \mathbf{A}_{k-i+2}$ for $i = 2, \ldots, k$.

The limiting spectral measure of $\mathbb{C}^{(k)}$

Proposition. (T.O.)

Fix $k \geq 2$. If $(\{\mathbf{A}_n^{(\ell)}\}; \ell = 1, \ldots, \lfloor \frac{k}{2} \rfloor + 1)$ are independent sequences of Wigner random matrices and for each n

$$\mathbb{C}_n^{(k)} := \mathbb{C}^{(k)}(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(\lfloor \frac{k}{2} \rfloor + 1)})$$

then

$$\mu_{\mathbb{C}_n^{(k)}} \xrightarrow{\omega} \nu_k \quad \text{as } n \to \infty \qquad a.s.$$

where

$$\nu_k := \begin{cases} \frac{k-2}{k}\, \gamma_{\frac{k-2}{k}} + \frac{2}{k}\, \gamma_{\frac{2k-2}{k}}, & k \text{ is even;} \\ \frac{k-1}{k}\, \gamma_{\frac{k-1}{k}} + \frac{1}{k}\, \gamma_{\frac{2k-1}{k}}, & k \text{ is odd.} \end{cases}$$

Examples

Case $k = 4$

$$\mu_{\mathbb{C}_n^{(4)}} \xrightarrow{\omega} \frac{1}{2}\,\gamma_{\frac{1}{2}} + \frac{1}{2}\,\gamma_{\frac{3}{2}} \quad \text{as } n \to \infty \qquad a.s.$$

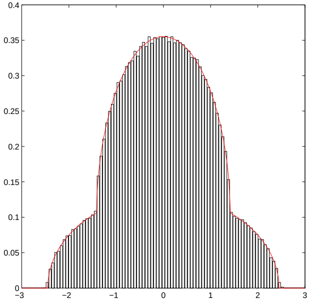

Figure: The limiting distribution of $\mu_{\mathbb{C}_n^{(4)}}$.

Examples

Case $k = 5$

$$\mu_{\mathbb{C}_n^{(5)}} \xrightarrow{\omega} \frac{4}{5}\,\gamma_{\frac{4}{5}} + \frac{1}{5}\,\gamma_{\frac{9}{5}} \quad \text{as } n \to \infty \qquad a.s.$$

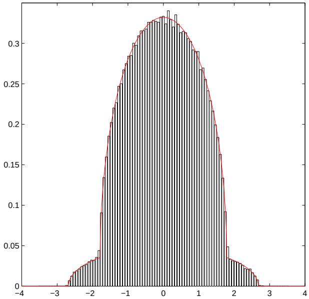

Figure: The limiting distribution of $\mu_{\mathbb{C}_n^{(5)}}$.

Outline

Existence Theorem

Theorem (T.O.)

Let $(\{\mathbf{A}_n^{(\ell)}\}; \ell = 1, \ldots, h)$ *be a family of independent sequences of* $n \times n$ *Wigner random matrices. For a fixed* $k \times k$ *Hermitian block-structure* $\mathbb{B}$ *and each* n*, define*

$$\mathbf{X}_n := \mathbb{B}(\mathbf{A}_n^{(1)}, \mathbf{A}_n^{(2)}, \ldots, \mathbf{A}_n^{(h)}).$$

Then there exists a non-random probability measure $\nu_{k,h,\mathbb{B}}$ *which depends only on* k, h *and the block structure* $\mathbb{B}$ *such that*

$$\mu_{\mathbf{X}_n} \xrightarrow{\omega} \nu_{k,h,\mathbb{B}} \quad \text{as } n \to \infty \qquad \text{a.s.}$$

Outline of the proof.

1. The proof is based on the method of moments.
2. Lemma. (T.O.) For $s = 1, 2, \ldots$

$$\int_{\mathbb{R}} x^s \mu_{\mathbf{X}_n}(dx) \to \int_{\mathbb{R}} x^s \mu_{\mathbf{X}}(dx) \quad \text{as } n \to \infty \qquad a.s.$$

   where ...
   - $\mathbf{X} = \mathbb{B}(\mathbf{a}_1, \ldots, \mathbf{a}_h)$
   - $\mathbf{a}_1, \ldots, \mathbf{a}_h$ are free semicircle random variables in some noncommutative probability space.
3. For $s = 1, 2, \ldots$

$$\int_{\mathbb{R}} x^s \mu_{\mathbf{X}_n}(dx) = \frac{1}{kn} Trace(\mathbf{X}_n^s)$$

4. $\mu_{\mathbf{X}}$ has a compact support in $\mathbb{R}$.

Thank you