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Double Quantum Groups and Iwasawa Decomposition

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The double quantum groups $\mathbb{C}_q[D(G)] = \mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ are the Hopf algebras underlying the complex quantum groups of which the simplest example is the quantum Lorentz group. They are nonstandard quantizations of the double group $G \times G$. We construct a corresponding quantized universal enveloping algebra $U_q(\mathfrak{d}(\mathfrak{g}))$ and prove that the pairing between $\mathbb{C}_q[D(G)]$ and $U_q(\mathfrak{d}(\mathfrak{g}))$ is nondegenerate. We analyze the representation theory of these $\mathbb{C}_q[D(G)]$, give a detailed version of the Iwasawa decomposition proved by Podles and Woronowicz for the quantum Lorentz group, and show that $\mathbb{C}_q[D(G)]$ is noetherian. Finally we outline how to construct more general nonstandard quantum groups using quantum double groups and their generalizations.

1. INTRODUCTION

A *double quantum group* is a Hopf algebra of the form $A \bowtie A$, where A is a braided Hopf algebra and $A \bowtie A$ denotes the generalized Drinfeld double constructed using the pairing defined by the braiding. This construction originated independently in Podles and Woronowicz's construction of the quantum Lorentz group and in early work of Majid on double cross products. Here we analyze this construction from an algebraic point of view with emphasis on the case when A is the standard quantum group $\mathbb{C}_q[G]$ for G a connected semisimple complex algebraic group. In this case the double $\mathbb{C}_q[D(G)] = \mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$ can be thought of as a nonstandard quantization of $\mathbb{C}[G \times G]$. One reason why the double is a particularly natural object is the existence of surjective Hopf algebra maps m: $\mathbb{C}_q[D(G)] \to \mathbb{C}_q[G]$ and θ: $\mathbb{C}_q[D(G)] \to U_q(\mathfrak{g})$. These maps are quantizations of the natural embeddings of G and its dual G_r into the double group $G \times G$ that occur in the analysis of the symplectic leaves of G and of dressing transformations on G [29, 21, 17].

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The main results we prove here are:

(1) The category of right $\mathbb{C}_q[D(G)]$-comodules is equivalent to the category of right $\mathbb{C}_q[G \times G]$-comodules as a braided rigid monoidal category.

(2) The natural pairing between $\mathbb{C}_q[D(G)]$ and its Fadeev–Reshetikhin–Takhtadjan (FRT) dual $U_q(\mathfrak{d}(\mathfrak{g}))$ is nondegenerate. Hence $U_q(\mathfrak{d}(\mathfrak{g}))$ has a category of finite dimensional left modules which is equivalent to the category of right $\mathbb{C}_q[D(G)]$-comodules and $\mathbb{C}_q[D(G)]$ is the restricted Hopf dual of $U_q(\mathfrak{d}(\mathfrak{g}))$ with respect to this category.

(3) The map $(m \otimes \theta)\Delta: \mathbb{C}_q[D(G)] \to \mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}}$ is injective. This is a generalization of the "Iwasawa decomposition" proved for the quantum Lorentz group in [27]. More precisely we show that there is a finite group Γ acting on $\mathbb{C}_q[G] \times U_q(\mathfrak{g})^{\mathrm{cop}}$ and an Ore set $\mathcal{S}$ of $\mathbb{C}_q[D(G)]$ such that the induced map from $\mathbb{C}_q[D(G)]_{\mathcal{S}}$ to $(\mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}})^{\Gamma}$ is an isomorphism.

(4) The algebra $\mathbb{C}_q[D(G)]$ is noetherian.

The constructions connected with the Iwasawa decomposition go back to work of Reshetikhin and Semenov-Tian-Shansky [28] on factorizable Hopf algebras. They prove that if A is finite dimensional and its dual A^* is factorizable, then there is an isomorphism from $A \bowtie A$ to $A \otimes (A^*)^{\mathrm{cop}}$ which is a Hopf algebra isomorphism for an appropriate twisted coalgebra structure on the latter (see also [25, 7.3]). For infinite dimensional algebras, the situation is more complicated, and in the first section we attempt to clarify exactly which parts of this theory still hold in this case. In the second section we look in detail at the case where $A = \mathbb{C}_q[G]$ and q is a complex parameter. Some care has to be taken in comparing these results to those in the literature where it is often stated that the Iwasawa map is an isomorphism. Such assertions either involve quantum groups in different settings (such as that of Podles and Woronowicz) or are heuristic assertions based on the finite dimensional case.

A key starting point is the observation by Doi and Takeuchi [10] that the double $A \bowtie A$ can be realized as a twist of the usual tensor product by a 2-cocycle (a construction dual to Drinfeld's gauge transformation [12]). From this result, assertion (1) is immediate and then (2) and the injectivity of the Iwasawa map follow by standard calculations. The second part of (3) (the identification of the image of the Iwasawa decomposition map) appears to be new even in the case $G = SL(2)$. The proof that $\mathbb{C}_q[D(G)]$ is noetherian uses the approach of Brown and Goodearl [4]. Since both $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ are homomorphic images of $\mathbb{C}[D(G)]$ this provides an interesting unified proof of the noetherianity of the standard quantum function algebra and the standard quantized enveloping algebra.

The algebra $\mathbb{C}_q[SL(2)]$ was introduced and studied (using different constructions) by Podles and Woronowicz in [27] and by Carow-Watamura, Schlieker, Scholl, and Watamura [5]. Further work on this and more general "complex quantum groups" was done by a variety of authors (see for instance [6, 30, 11]). In [22], Majid pointed out that these algebras could be constructed using the quantum double construction that we consider here. This approach was also taken up in [7]. This article is, in a sense, a continuation of the approach of [22] and [7], with the important distinction that we do not consider the $*$-algebra structure. All of the above papers are concerned with quantizing the algebra of complex valued coordinate functions on a simple complex algebraic group G, considered as a real Lie group. Thus the relevant notion is a complex Hopf algebra equipped with a $*$-operation. Here we consider these algebras rather as quantifications of the algebra of regular functions on the double complex algebraic group $G \times G$. However, the underlying Hopf algebra is the same and many of the basic structural results are relevant in both contexts.

We also show how to apply the same twisting technique to construct some new nonstandard quantum groups. Recall that the Poisson group structures on a semisimple Lie group G given by a solution of the modified classical Yang–Baxter equation, have been classified by Belavin and Drinfeld [2]. They are given (in part) by triples of the form $(\tau, \mathbf{B}_1, \mathbf{B}_2)$, where $\mathbf{B}_i$ are subsets of the base $\mathbf{B}$ of the root system for $\mathfrak{g}$ and τ is a bijection from $\mathbf{B}_1$ to $\mathbf{B}_2$ satisfying certain conditions. In the special case where $\mathbf{B}_1$ and $\mathbf{B}_2$ are "disjoint" we construct quantizations of the associated Poisson group. This construction is closely related to the construction of nonstandard quantum groups by Fronsdal and Galindo using the universal T-matrix [13].

The Hopf algebra notation that we use is generally the standard notation used in Sweedler's book except that we remove the parenthesis from the "Sweedler notation" writing $\Delta(x) = \sum x_1 \otimes x_2$ rather than $\sum x_{(1)} \otimes x_{(2)}$. We use m, Δ, ϵ, and S indiscriminately for the multiplication, comultiplication, counit, and antipode in any Hopf algebra.

2. BRAIDED HOPF ALGEBRAS AND THEIR DOUBLES

An appropriately general setting in which to view the basic constructions is that of a *braided Hopf algebra*. Braided Hopf algebras are also known as dual quasitriangular, coquasitriangular, or cobraided Hopf algebras. They play the role (roughly) of the dual of a quasitriangular Hopf algebra and include all of the standard, multiparameter, and nonstandard quantizations of semisimple algebraic groups. The notation follows closely that of Doi and Takeuchi but many of the ideas occur in the work of Majid [22, 24, 26] and Larson and Towber [20]. In their dual form these results go back to Drinfeld [12] and Reshetikhin and Semenov-Tian-Shansky [28].

Let A and U be a pair of Hopf algebras. A *skew pairing* on the ordered pair (A, U) is a bilinear form τ satisfying

(1) $\tau(bc, u) = \sum \tau(b, u_1)\tau(c, u_2)$;

(2) $\tau(b, uv) = \sum \tau(b_1, v)\tau(b_2, u)$;

(3) $\tau(1, u) = \epsilon(u)$; $\tau(a, 1) = \epsilon(a)$.

A bilinear form on the pair (A, U) is said to be invertible if it is invertible as an element of $(A \otimes U)^*$. Any skew pairing τ is invertible with inverse given by

$$\tau^{-1}(a, u) = \tau(S(a), u).$$

For any bilinear form τ we denote by τ_{21} the pairing given by $\tau_{21}(a, b) = \tau(b, a)$. Note that if τ is a skew pairing, then so is τ_{21}^{-1}.

Given a skew pairing on a pair of Hopf algebras (A, U), one may construct a new Hopf algebra, the *quantum double* $A \bowtie_\tau U$ (or just $A \bowtie U$). As a coalgebra the double is isomorphic to $A \otimes U$. The multiplication is given by the rule

$$(1 \otimes u)(a \otimes 1) = \sum \tau(a_1, u_1)a_2 \otimes u_2\tau^{-1}(a_3, u_3)$$

or, equivalently,

$$\sum (1 \otimes u_1)(a_1 \otimes 1)\tau(a_2, u_2) = \sum \tau(a_1, u_1)a_2 \otimes u_2.$$

A *braided Hopf algebra* is a Hopf algebra A together with an invertible skew pairing β on (A, A) such that

$$\sum b_1 a_1 \beta(a_2, b_2) = \sum \beta(a_1, b_1)a_2 b_2.$$

The pairing β is called a *braiding* on A. Note that the antipode in a braided algebra is always bijective [9, 24]. Given a braided Hopf algebra A we may form the double Hopf algebra $A \bowtie_\beta A$.

A 2-*cocycle* on a Hopf algebra A is an invertible pairing $\sigma\colon A \otimes A \to k$ such that for all x, y, and z in A,

$$\sum \sigma(x_1, y_1)\sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\sigma(x, y_2 z_2)$$

and $\sigma(1, 1) = 1$. Given a 2-cocycle σ on a Hopf algebra A, one can twist the multiplication to get a new Hopf algebra A_σ [10]. The new multiplication is given by

$$x \cdot y = \sum \sigma(x_1, y_1)x_2 y_2 \sigma^{-1}(x_3, y_3).$$

The comultiplication in A_σ remains the same. In particular, A and A_σ are isomorphic as coalgebras. This construction is essentially the dual of Drinfeld's gauge transformation [12].

Recall that the category of right comodules over a braided Hopf algebra has a natural structure of a braided rigid monoidal category [20, 25]. A 2-cocycle σ on a braided Hopf algebra can be used to define an equivalence of such categories between $\operatorname{Comod} A$ and $\operatorname{Comod} A_\sigma$ when A_σ is equipped with the braiding defined below.

THEOREM 2.1. *Let A be a braided Hopf algebra with braiding β. Let σ be a 2-cocycle on A. Let A_σ be the twisted Hopf algebra defined above. Then $\sigma_{21} * \beta * \sigma^{-1}$ (convolution product) is a braiding on A_σ. Moreover the categories of comodules over A and A_σ are equivalent as rigid braided monoidal categories.*

Proof. See [23]. ∎

Doi and Takeuchi [10] observed that the quantum double may be constructed from the tensor product by twisting by a 2-cocycle. Let τ be an invertible skew pairing between the Hopf algebras B and H. Then the bilinear form $[\tau]$ defined on $B \otimes H$ by

$$[\tau](b \otimes g, c \otimes h) = \epsilon(b)\epsilon(h)\tau(c, g)$$

is a 2-cocycle and the twisted Hopf algebra $(B \otimes H)_{[\tau]}$ is isomorphic as a Hopf algebra to the quantum double $B \bowtie_\tau H$ [10, Prop. 2.2]. In particular, if (A, β) is a braided Hopf algebra, then the double $A \bowtie_\beta A$ is isomorphic to $(A \otimes A)_{[\beta]}$.

There are a number of different ways of defining a braiding on $A \bowtie A$. Recall that if β is a braiding on A, then so is β_{21}^{-1}. The tensor algebra $A \otimes A$ can therefore be made into a braided algebra in a number of different ways using these two braidings. It turns out that the appropriate choice is the braiding given by β on the first component and β_{21}^{-1} on the second component which we shall denote by (β, β_{21}^{-1}). Using the above theorem we deduce that

$$\gamma = [\beta]_{21} * (\beta, \beta_{21}^{-1}) * [\beta]^{-1}$$

is a braiding on $A \bowtie A$ (cf. [22, Proposition 2]). This braiding makes the category of right comodules into a braided category.

THEOREM 2.2. *Let (A, β) be a braided Hopf algebra. Then the double $A \bowtie A$ has a braiding γ given by*

$$\gamma(a \otimes b, a' \otimes b') = \beta(a, a'_1 b'_1)\beta^{-1}(a'_2 b'_2, b).$$

The category $\operatorname{Comod} A \bowtie A$ *is equivalent as a braided rigid monoidal category to* $\operatorname{Comod} A \otimes A$.

There are two important Hopf algebra homomorphisms associated to the double $A \bowtie A$ of a braided Hopf algebra A. The first is the multiplication map [10, Prop. 3.1]

$$m\colon A \bowtie A \to A.$$

The second is the map

$$\theta\colon A \bowtie A \to (A^\circ)^{\mathrm{cop}}$$

defined by [10, Theorem 3.2]

$$\theta(x \otimes y) = \beta(x, -)\beta^{-1}(-, y).$$

Notice that $\theta = m(l^+ \otimes l^-)$, where $l^\pm\colon A \to (A^\circ)^{\mathrm{cop}}$ are the maps $l^+(x) = \beta(x, -)$ and $l^-(y) = \beta^{-1}(-, y)$. We denote the image of θ in A° by $U(A)$ and refer to it as the *FRT-dual* of A. Recall that $U(A)$ is said to be dense in A° if the pairing between A and $U(A)$ is nondegenerate. Combining the above maps via comultiplication yields an algebra map

$$\xi = (\theta \otimes m)\Delta\colon A \bowtie A \to U(A)^{\mathrm{cop}} \otimes A.$$

Given the braiding γ on $A \bowtie A$ we have an associated FRT-dual $U(A \bowtie A)$ and maps $l^\pm\colon A \bowtie A \to U(A \bowtie A)$. For any Hopf algebra B we shall denote by $\langle -, - \rangle$ the pairing between B° and B. The maps m and θ have duals $m^*\colon U(A) \to (A \bowtie A)^\circ$ and $\theta^*\colon A^{\mathrm{op}} \to (A \bowtie A)^\circ$ given by $\langle m^*(u), b \otimes b' \rangle = \langle u, m(b \otimes b') \rangle = \langle u, bb' \rangle$ and $\langle \theta^*(a), b \otimes b' \rangle = \langle \theta(b \otimes b'), a \rangle = \sum \beta(b, a_1)\beta^{-1}(a_2, b')$.

LEMMA 2.3. *Consider the maps $l^\pm\colon A \bowtie A \to U(A \bowtie A)$. Then $l^+ = m^*\theta$ and $l^- = S\theta^* m$. In particular, the images of both m^* and θ^* are contained in $U(A \bowtie A)$.*

Proof. Observe that

$$\gamma(a \otimes b, a' \otimes b') = \langle \theta(a \otimes b), m(a' \otimes b') \rangle = \langle m^*\theta(a \otimes b), a' \otimes b' \rangle.$$

Similarly,

$$\begin{aligned} \gamma^{-1}(a \otimes b, a' \otimes b') &= \langle \theta(a \otimes b), S^{-1}m(a' \otimes b') \rangle \\ &= \langle S\theta^* m(a' \otimes b'), a \otimes b \rangle. \quad \blacksquare \end{aligned}$$

Recall that the usual pairing between $U(A)$ and A becomes a skew pairing between $U(A)$ and A^{op}, allowing us to form the double $U(A) \bowtie A^{\mathrm{op}}$.

THEOREM 2.4. *The map*

$$\zeta = m(m^* \otimes \theta^*) \colon U(A) \bowtie A^{\mathrm{op}} \to U(A \bowtie A)$$

is a surjective Hopf algebra map.

Proof. The lemma implies that the image of ζ is precisely $U(A \bowtie A)$. The fact that ζ is a Hopf algebra map is more or less well known (see for instance [22]). It follows from the formula

$$\sum \theta^*(a_1) m^*(u_1) \langle u_2, a_2 \rangle = \sum \langle u_1, a_1 \rangle m^*(u_2) \theta^*(a_2),$$

which may be verified directly. ∎

COROLLARY 2.5. *If $U(A \bowtie A)$ is dense in $(A \bowtie A)^\circ$, then the map*

$$\xi = (m \otimes \theta)\Delta \colon A \bowtie A \to A \otimes U(A)^{\mathrm{cop}}$$

is injective.

Proof. Notice that the pairing between $A \bowtie A$ and $U(A) \bowtie A^{\mathrm{op}}$ induced from ζ is the same as that induced from the map ξ. Thus

$$\operatorname{Ker} \xi \subset \left(U(A) \bowtie A^{\mathrm{op}}\right)^{\perp} \subset U(A \bowtie A)^{\perp}.$$

The density of $U(A \bowtie A)$ implies that $U(A \bowtie A)^{\perp} = 0$, so $\operatorname{Ker} \xi = 0$ also. ∎

We now give an alternative construction of the braiding of $A \bowtie A$. This approach is analogous to Drinfeld's construction of a universal R-matrix for the Drinfeld double $H^* \bowtie H$. We follow the reformulation of this construction given in [14, 19]. This construction is also needed later in defining the braiding on the standard quantum groups. Let A and B be Hopf algebras and let σ be a skew pairing between A and B. Define Hopf algebra maps

$$\Phi_1 \colon A^{\mathrm{cop}} \to B^\circ, \qquad \Phi_2 \colon B^{\mathrm{op}} \to A^\circ$$

by $\Phi_1(a) = \sigma(a, -)$ and $\Phi_2(b) = \sigma(-, b)$. If the pairing σ is nondegenerate these maps will be injective. Henceforth we shall assume that this is the case. Now suppose that we have a Hopf pairing ϕ between C and $A \bowtie B$ which identifies C with a subalgebra of the dual of $A \bowtie B$. There are Hopf algebra maps

$$\theta_1 \colon C \to B^\circ, \qquad \theta_2 \colon C \to A^\circ$$

defined by $\langle \theta_1(c), b \rangle = \phi(c, 1 \otimes b)$ and $\langle \theta_2(c), a \rangle = \phi(c, a \otimes 1)$, respectively.

In order to construct a braiding on C we need to assume that $\operatorname{Im} \theta_i \subset \operatorname{Im} \Phi_i$ for $i = 1$ and 2. In this case we can construct maps

$$\psi_1 = \Phi_1^{-1}\theta_1 : C \to A^{\mathrm{cop}}, \qquad \psi_2 = \Phi_2^{-1}\theta_2 : C \to B^{\mathrm{op}}.$$

THEOREM 2.6. *The form* $\beta \in (C \otimes C)^*$ *defined by*

$$\beta(x, y) = \sigma(\psi_1(x), \psi_2(y))$$

is a braiding on C.

Let $\pi: A \bowtie B \to C^*$ *be the natural map and let* $l^{\pm}: C \to U(C)^{\mathrm{cop}}$ *be the maps described above. Then* $l^+ = \pi\psi_1$ *and* $l^- = S\pi\psi_2$.

Proof. See [19, 9.4.6]. ∎

We now apply this result in the case where σ is the natural skew pairing between $U(A)$ and A^{op} for some braided Hopf algebra A and C is $A \bowtie A$. When $U(A \bowtie A)$ is dense in $(A \bowtie A)^{\circ}$, this produces a braiding on $A \bowtie A$ which we will denote by γ'.

THEOREM 2.7. *The braidings* γ *and* γ' *coincide*.

Proof. The pairing between $U(A) \bowtie A^{\mathrm{op}}$ and $A \bowtie A$ is given by

$$\begin{aligned} \langle u \otimes c, a \otimes b \rangle &= \langle m^*(u), a_1 \otimes b_1 \rangle \langle \theta^*(c), a_2 \otimes b_2 \rangle \\ &= \langle u, a_1 b_1 \rangle \langle \theta(a_2 \otimes b_2), c \rangle. \end{aligned}$$

So

$$\langle \theta_1(a \otimes b), c \rangle = \langle 1 \times c, a \otimes b \rangle = \langle \theta(a \otimes b), c \rangle = \langle \Phi_1 \theta(a \otimes b), c \rangle.$$

Hence $\psi_1 = \Phi_1^{-1}\theta_1 = \theta$. Similarly,

$$\langle \theta_2(a \otimes b), u \rangle = \langle u \otimes 1, a \otimes b \rangle = \langle u, m(a \otimes b) \rangle = \langle \Phi_2 m(a \otimes b), u \rangle.$$

Thus $\psi_2 = m$. Hence

$$\gamma'(a \otimes b, a' \otimes b') = \langle \theta(a \otimes b), m(a' \otimes b') \rangle = \gamma(a \otimes b, a' \otimes b')$$

as required. ∎

This theorem says that the braiding γ is essentially the same object as the universal T-matrix of [28] and [13].

There is a natural algebra embedding of $U(A \bowtie A)$ into $U(A) \otimes U(A)$ given by

$$\chi(u) = \sum f(u_1) \otimes f'(u_2),$$

where $f(u)$ and $f'(u)$ denote the restriction of u to the first and second copies of A, respectively. Of particular importance is the composition of χ and ζ.

LEMMA 2.8. 1. *For all $a \in A$, $\chi\theta^*(a) = \sum l^- S(a_1) \otimes l^+ S(a_2)$.*

2. *For all $u \in U(A)$, $\chi m^*(u) = \sum u_1 \otimes u_2 = \Delta(u)$.*

Proof. Notice that

$$\begin{aligned}\langle \theta^*(a), b \otimes c \rangle = \langle \theta(b \otimes c), a \rangle &= \beta(b, a_1)\beta^{-1}(a_2, c)\\ &= \beta^{-1}(b, S(a_1))\beta(S(a_2), c)\\ &= \langle l^- S(a_1), b\rangle\langle l^+ S(a_2), c\rangle\end{aligned}$$

This proves the first assertion. The second assertion is proved similarly. ∎

3. DOUBLE QUANTUM GROUPS AND THE QUANTUM LORENTZ GROUP

We now define the double quantum group associated to a connected complex semisimple algebraic group G. We continue to use the notation of [18]. For the convenience of the reader we recall briefly the relevant details.

Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathbf{R}$ the associated root system, $\mathbf{B} = \{\alpha_1, \ldots, \alpha_n\}$ a basis of $\mathbf{R}$, $\mathbf{R}_+$ the set of positive roots, and W the Weyl group. We denote by $\mathbf{P}$ and $\mathbf{Q}$ the lattices of weights and roots, respectively, and by $\mathbf{P}^+$ the set of dominant integral weights. Let H be a maximal torus of G with Lie algebra $\mathfrak{h}$ and denote by $\mathbf{L}$ the character group of H, which we shall identify with a sublattice of $\mathbf{P}$ containing $\mathbf{Q}$. Let $(-, -)$ be the Killing form on $\mathfrak{h}^*$ and set $d_i = (\alpha_i, \alpha_i)/2$. Set $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \mathbf{R}_+} \mathfrak{g}_{\pm\alpha}$ and $\mathfrak{b}^{\pm} = \mathfrak{h} \oplus \mathfrak{n}^{\pm}$.

Let $q \in \mathbb{C}^*$ and assume that *q is not a root of unity*. Since we need to consider rational powers of q we adopt the following convention. Pick $\hbar \in \mathbb{C}$ such that $q = e^{-\hbar/2}$ and define $q^m = e^{-m\hbar/2}$ for all $m \in \mathbb{Q}$. We set $q_i = q^{d_i}$ for $i = 1, \ldots, n$. Denote by U^0 the group algebra of $\mathbf{L}$,

$$U^0 = \mathbb{C}[k_\lambda;\ \lambda \in \mathbf{L}], \qquad k_0 = 1, \qquad k_\lambda k_\mu = k_{\lambda+\mu}.$$

Set $k_i = k_{\alpha_i}$, $1 \leq i \leq n$. The standard quantized enveloping algebra associated to this data is the Hopf algebra

$$U_q(\mathfrak{g}) = U^0[e_i, f_i; 1 \leq i \leq n]$$

with defining relations:

$$k_\lambda e_j k_\lambda^{-1} = q^{(\lambda, \alpha_j)} e_j, \qquad k_\lambda f_j k_\lambda^{-1} = q^{-(\lambda, \alpha_j)} f_j,$$

$$e_i f_j - f_j e_i = \delta_{ij}\left(k_i - k_i^{-1}\right)/\left(q^{d_i} - q^{-d_i}\right),$$

and the quantum Serre relations as given in [18]. The Hopf algebra structure is given by

$$\Delta(k_\lambda) = k_\lambda \otimes k_\lambda, \qquad \epsilon(k_\lambda) = 1, \qquad S(k_\lambda) = k_\lambda^{-1},$$

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i,$$

$$\epsilon(e_i) = \epsilon(f_i) = 0, \qquad S(e_i) = -k_i^{-1} e_i, \qquad S(f_i) = -f_i k_i.$$

Notice that $U_q(\mathfrak{g})$ depends on G rather than $\mathfrak{g}$. Thus the notation is a little ambiguous.

We define subalgebras of $U_q(\mathfrak{g})$ as follows:

$$U_q(\mathfrak{b}^+) = U^0[e_i; 1 \le i \le n], \qquad U_q(\mathfrak{b}^-) = U^0[f_i; 1 \le i \le n].$$

Notice that U^0 and $U_q(\mathfrak{b}^\pm)$ are Hopf subalgebras of $U_q(\mathfrak{g})$.

Let M be a left $U_q(\mathfrak{g})$-module. An element $x \in M$ is said to have *weight* $\mu \in \mathbf{L}$ if $k_\lambda x = q^{(\lambda, \mu)} x$ for all $\lambda \in \mathbf{L}$; we denote by M_μ the subspace of elements of weight μ. It is known that the category of finite dimensional $U_q(\mathfrak{g})$-modules is a completely reducible braided rigid monoidal category. Set $\mathbf{L}^+ = \mathbf{L} \cap \mathbf{P}^+$ and recall that for each $\lambda \in \mathbf{L}^+$ there exists a finite dimensional simple module of highest weight λ, denote by $L(\lambda)$. One has $L(\lambda)^* \cong L(w_0 \lambda)$, where w_0 is the longest element of W. Let $\mathcal{C}_q$ be the subcategory of finite dimensional $U_q(\mathfrak{g})$-modules consisting of finite direct sums of $L(\lambda)$, $\lambda \in \mathbf{L}^+$. The category $\mathcal{C}_q$ is closed under tensor products and the formation of duals.

Let M be an object of $\mathcal{C}_q$. Then $M = \bigoplus_{\mu \in \mathbf{L}} M_\mu$. For $f \in M^*$ and $v \in M$ we define the coordinate function $c_{f,v} \in U_q(\mathfrak{g})^*$ by

$$\forall u \in U_q(\mathfrak{g}), \qquad c_{f,v}(u) = \langle f, uv \rangle,$$

where $\langle\ ,\ \rangle$ is the duality pairing. The quantized function algebra $\mathbb{C}_q[G]$ is the restricted dual of $U_q(\mathfrak{g})$ with respect to $\mathcal{C}_q$. That is,

$$\mathbb{C}_q[G] = \mathbb{C}\left[c_{f,v}; v \in M, f \in M^*, M \in \operatorname{obj}(\mathcal{C}_q)\right].$$

The algebra $\mathbb{C}_q[G]$ is a Hopf algebra. If $\{v_1, \dots, v_s; f_1, \dots, f_s\}$ is a dual basis for M one has

$$\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f_i,v}, \qquad \epsilon(c_{f,v}) = \langle f, v \rangle, \qquad S(c_{f,v}) = c_{v,f}. \tag{3.1}$$

Notice that we may assume that $v_j \in M_{\nu_j}$, $f_j \in M^*_{-\nu_j}$. When $v \in L(\lambda)_\mu$ and $f \in L(\lambda)^*_{-\nu}$ we denote the element $c_{f,v}$ by $c^\lambda_{-\nu,\mu}$. Although convenient this notation is a little ambiguous and some care has to be taken in interpreting the standard formulas such as $\Delta(c^\lambda_{\nu,\mu}) = c^\lambda_{\nu,\mu_i} \otimes c^\lambda_{-\mu_i,\mu}$ and $S(c^\lambda_{\nu,\mu}) = c^{-w_0\lambda}_{\mu,\nu}$.

Recall that the Rosso form $\phi(\ ,\ )$ defines a skew pairing on $(U_q(\mathfrak{b}^-), U_q(\mathfrak{b}^+))$. Consider the induced maps

$$\Phi_1 \colon U_q(\mathfrak{b}^-)^{\mathrm{cop}} \to U_q(\mathfrak{b}^+)^\circ, \qquad \Phi_2 \colon U_q(\mathfrak{b}^+)^{\mathrm{op}} \to U_q(\mathfrak{b}^-)^\circ$$

and the maps

$$\theta_1 \colon \mathbb{C}_q[G] \to U_q(\mathfrak{b}^+)^\circ, \qquad \theta_2 \colon \mathbb{C}_q[G] \to U_q(\mathfrak{b}^-)^\circ$$

as defined above. By [18, Prop. 4.6], we have that $\operatorname{Im}\theta_i = \operatorname{Im}\Phi_i$. Hence the Rosso form induces a braiding on $\mathbb{C}_q[G]$ defined by

$$\beta(x, y) = \phi(\psi_1(x), \psi_2(y)),$$

where $\psi_i = \Phi_i^{-1}\theta_i$.

Definition. We define $\mathbb{C}_q[D(G)]$ to be the double quantum group $\mathbb{C}_q[G] \bowtie \mathbb{C}_q[G]$.

Example 1. In the case when $G = SL(2,\mathbb{C})$, the double quantum group $\mathbb{C}_q[D(SL(2))]$ is the Hopf algebra underlying the quantum Lorentz group. It is often denoted by $SL_q(2,\mathbb{C})$. As an algebra it is generated by the elements a, b, c, d and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ subject to the following relations:

$$ba = qab, \qquad ca = qac, \qquad db = qbd, \qquad dc = qcd,$$
$$cb = bc, \qquad ad - da = (q - q^{-1})bc, \qquad da - qbc = 1,$$

$$\tilde{b}\tilde{a} = q\tilde{a}\tilde{b}, \qquad \tilde{c}\tilde{a} = q\tilde{a}\tilde{c}, \qquad \tilde{d}\tilde{b} = q\tilde{b}\tilde{d}, \qquad \tilde{d}\tilde{c} = q\tilde{c}\tilde{d},$$
$$\tilde{c}\tilde{b} = \tilde{b}\tilde{c}, \qquad \tilde{a}\tilde{d} - \tilde{d}\tilde{a} = (q - q^{-1})\tilde{b}\tilde{c}, \qquad \tilde{d}\tilde{a} - q\tilde{b}\tilde{c} = 1,$$
$$a\tilde{a} = \tilde{a}a, \qquad qa\tilde{b} = \tilde{b}a, \qquad a\tilde{c} = \tilde{c}a + (q - q^{-1})c\tilde{a},$$
$$a\tilde{d} + (q - q^{-1})c\tilde{b} = \tilde{d}a, \qquad qb\tilde{a} = \tilde{a}b = (q - q^{-1})\tilde{b}a,$$
$$b\tilde{b} = \tilde{b}b, \qquad b\tilde{c} + (q - q^{-1})d\tilde{a} = (q - q^{-1})\tilde{d}a + \tilde{a}b,$$
$$b\tilde{d} + (q - q^{-1})d\tilde{b} = q\tilde{d}b,$$

$$c\tilde{a} = q\tilde{a}c, \qquad c\tilde{b} = \tilde{b}c, \qquad c\tilde{c} = \tilde{c}c, \qquad qc\tilde{d} = \tilde{d}c,$$
$$d\tilde{a} = \tilde{a}d + (q - q^{-1})\tilde{b}c, \qquad d\tilde{b} = q\tilde{b}d,$$
$$qd\tilde{c} = \tilde{c}d + (q - q^{-1})\tilde{d}c, \qquad d\tilde{d} = \tilde{d}d.$$

We being by stating explicitly Theorem 2.2 for this case.

THEOREM 3.1. *The category* $\operatorname{Comod} \mathbb{C}_q[D(G)]$ *is equivalent as a braided rigid monoidal category to* $\operatorname{Comod} \mathbb{C}_q[G \times G]$.

The equivalence of the comodule categories in the case of the quantum Lorentz group was proved by Podles and Woronowicz in their original paper [27]. To our knowledge no more general result has appeared in the literature, although this result is more or less implicit in some of Majid's work.

DEFINITION. Define $U_q(\mathfrak{d}(\mathfrak{g}))$ to be the FRT-dual $U(\mathbb{C}_q[D(G)])$.

Notice again that $U_q(\mathfrak{d}(\mathfrak{g}))$ depends not only on $\mathfrak{g}$ but also on the choice of G. Theorem 2.4 gives a weak version of Iwasawa decomposition for $U_q(\mathfrak{d}(\mathfrak{g}))$.

THEOREM 3.2. *The map*

$$\zeta = m(m^* \otimes \theta^*) \colon U_q(\mathfrak{g}) \bowtie \mathbb{C}_q[G]^{\mathrm{op}} \to U_q(\mathfrak{d}(\mathfrak{g}))$$

is an epimorphism of Hopf algebras.

We conjecture that this map ζ is in fact an isomorphism.

We now show that the pairing between $\mathbb{C}_q[D(G)]$ and $U_q(\mathfrak{d}(\mathfrak{g}))$ is nondegenerate. We need a detailed description of the maps $l^{\pm}$. Note that Theorem 2.6 implies that

$$\operatorname{Ker} l^+ = \left\{ c_{f,v} \mid f \in \left(U_q(\mathfrak{b}^+) v \right)^{\perp} \right\}$$

and

$$\operatorname{Ker} l^- = \left\{ c_{f,v} \mid f \in \left(U_q(\mathfrak{b}^-) v \right)^{\perp} \right\}$$

Hence $c^{\lambda}_{-\mu,\nu} \in \operatorname{Ker} l^+$ if $\mu - \nu$ is not a nonnegative integer combination of positive roots. Similarly $c^{\lambda}_{-\mu,\nu} \in \operatorname{Ker} l^-$ if $\nu - \mu$ is not a nonnegative integer combination of positive roots.

Define

$$U^+ = \mathbb{C}[e_i \mid 1 \le i \le n], \qquad U^- = \mathbb{C}[f_i \mid 1 \le i \le n]$$

and for all $\beta \in \mathbf{Q}$ set

$$U^{\pm}_{\beta} = \left\{ u \in U^{\pm} \mid k_{\lambda} u k_{-\lambda} = q^{(\lambda,\beta)} u \right\}.$$

LEMMA 3.3. *Let* $\lambda \in \mathbf{L}^+$ *and let* $\mu \in \mathbf{L}$.

(1) *There exists a unique* $y \in U^-_{\mu-\lambda}$ *such that* $l^+(c^{\lambda}_{-\lambda,\mu}) = y k_{-\mu}$.

(2) *There exists a unique* $x \in U^+_{\lambda-\mu}$ *such that* $l^-(c^{\lambda}_{-\mu,\lambda}) = x k_{\mu}$.

Proof. See [18] or [19, 9.2.11]. ∎

Consider the natural functor from right $\mathbb{C}_q[D(G)]$-comodules to left $U_q(\mathfrak{d}(\mathfrak{g}))$-modules. For a right $\mathbb{C}_q[D(G)]$-comodule V, we define an action of $U_q(\mathfrak{d}(\mathfrak{g}))$ by

$$u.v = \sum \langle v_1, u \rangle v_0$$

for $v \in V$ and $u \in U_q(\mathfrak{d}(\mathfrak{g}))$.

The proof of the following theorem generalizes the proof of an analogous result for the quantum Lorentz group given in [30].

THEOREM 3.4. *Let V be a simple right $\mathbb{C}_q[D(G)]$-comodule and consider V as a left $U_q(\mathfrak{d}(\mathfrak{g}))$-module. Then V is simple.*

Proof. By Theorem 3.1 any irreducible $\mathbb{C}_q[D(G)]$-comodule is of the form $V = L(\nu) \otimes L(\nu')$, where $\nu, \nu' \in \mathbf{L}^+$. The action of $U_q(\mathfrak{d}(\mathfrak{g}))$ on V is then given by the map χ and the usual action of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. Suppose that $\lambda \in \mathbf{L}^+$ and $\mu \in \mathbf{L}$. It follows from Lemma 2.8 that, for any $c \in \mathbb{C}_q[G]$,

$$\chi\theta^*(c) = l^- S(c_1) \otimes l^+ S(c_2).$$

Hence

$$\chi\theta^*\left(c^{\lambda}_{-\mu,\lambda}\right) = \sum l^- S\left(c^{\lambda}_{-\mu,\nu}\right) \otimes l^+ S\left(c^{\lambda}_{-\nu,\lambda}\right) = l^- S\left(c^{\lambda}_{-\mu,\lambda}\right) \otimes l^+ S\left(c^{\lambda}_{-\lambda,\lambda}\right)$$

and similarly

$$\chi\theta^*\left(c^{\lambda}_{-\lambda,\mu}\right) = l^- S\left(c^{\lambda}_{-\lambda,\lambda}\right) \otimes l^+ S\left(c^{\lambda}_{-\lambda,\mu}\right)$$

(cf. [19, 9.2.14]). In particular Lemma 3.3 yields

$$\chi\theta^*\left(c^{\lambda}_{-\lambda,\lambda}\right) = k_{-\lambda} \otimes k_{\lambda}.$$

Thus the image of $\chi\theta^*$ contains an "antidiagonal" copy of the subalgebra $\mathbb{C}[k_\lambda \mid \lambda \in \mathbf{L}^+]$ of U^0. On the other hand, the image under m^* of U^0 is a "diagonal" copy of U^0. It follows that any $U_q(\mathfrak{d}(\mathfrak{g}))$-submodule V' of V is a sum of its $U^0 \otimes U^0$-weight spaces. Now, applying Lemma 3.3 again for $\mu = \lambda - \alpha_j$ yields (up to a scalar factor)

$$\chi\theta^*\left(c^{\lambda}_{-\mu,\lambda}\right) = e_j k_{-\lambda} \otimes k_{\lambda},$$

$$\chi\theta^*\left(c^{\lambda}_{-\lambda,\mu}\right) = k_{-\lambda} \otimes f_j k_{\lambda}.$$

Hence if V' is a nonzero submodule of V it must contain a weight vector $v_+ \otimes v'_-$, where v'_- is a highest weight vector of $V(\nu')$ and v_+ is a lowest weight vector of $V(\nu)$. However, this element generates V as a $U_q(\mathfrak{g})$-module via the diagonal action. Thus $V' = V$ as required. ∎

The theorem yields a form of Peter–Weyl theorem linking $\mathbb{C}_q[D(G)]$ and $U_q(\mathfrak{d}(\mathfrak{g}))$. Denote by $\tilde{\mathcal{C}}_q$ the full subcategory of $U_q(\mathfrak{d}(\mathfrak{g}))$-Mod consisting of direct sums of modules of the form $L(\nu) \otimes L(\nu')$. Since $\tilde{\mathcal{C}}_q$ is closed under tensor products, duals, and direct sums we may form the restricted dual of $U_q(\mathfrak{d}(\mathfrak{g}))$ with respect to $\tilde{\mathcal{C}}_q$. This is the Hopf algebra of coordinate functions

$$U_q(\mathfrak{d}(\mathfrak{g}))^{\circ}_{\tilde{\mathcal{C}}_q} = \mathbb{C}\Big[c_{f,v};\, v \in M,\, f \in M^*,\, M \in \operatorname{obj}\big(\tilde{\mathcal{C}}_q\big)\Big].$$

COROLLARY 3.5. *The pairing between $\mathbb{C}_q[D(G)]$ and $U_q(\mathfrak{d}(\mathfrak{g}))$ is nondegenerate. The categories $\mathcal{C}_q$ and $\tilde{\mathcal{C}}_q$ are equivalent and $\mathbb{C}_q[D(G)]$ is the restricted dual of $U_q(\mathfrak{d}(\mathfrak{g}))$ with respect to $\tilde{\mathcal{C}}_q$.*

Proof. For $\nu \in \mathbf{L}^+$ let $C(L(\nu)) \subset \mathbb{C}_q[G]$ be the subcoalgebra of coordinate functions on $L(\nu)$. Then

$$\mathbb{C}_q[D(G)] = \bigoplus_{\nu, \nu' \in \mathbf{L}^+} C(L(\nu)) \otimes C(L(\nu')).$$

Similarly let $C(L(\nu) \otimes L(\nu'))$ be the subcoalgebra of $U_q(\mathfrak{d}(\mathfrak{g}))^*$ of coordinate functions on $L(\nu) \otimes L(\nu')$. Then

$$U_q(\mathfrak{d}(\mathfrak{g}))^{\circ}_{\tilde{\mathcal{C}}_q} = \bigoplus_{\nu, \nu' \in \mathbf{L}^+} C(L(\nu) \otimes L(\nu')).$$

Let η: $\mathbb{C}_q[D(G)] \to U_q(\mathfrak{d}(\mathfrak{g}))^*$ be the natural map. Then since $L(\nu) \otimes L(\nu')$ is simple η maps $C(L(\nu)) \otimes C(L(\nu'))$ isomorphically onto $C(L(\nu) \otimes L(\nu'))$. Hence η is an isomorphism from $\mathbb{C}_q[D(G)]$ to $U_q(\mathfrak{d}(\mathfrak{g}))^{\circ}_{\tilde{\mathcal{C}}_q}$. ∎

In particular this implies that the intersection of the annihilators of the simple modules $L(\nu) \otimes L(\nu')$ is zero. Hence $U_q(\mathfrak{d}(\mathfrak{g}))$ is semiprimitive and residually finite.

Applying Corollary 2.5 yields the following version of Iwasawa decomposition for $\mathbb{C}_q[D(G)]$.

COROLLARY 3.6. *The map*

$$\xi = (m \otimes \theta)\Delta\colon \mathbb{C}_q[D(G)] \to \mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}}$$

is injective.

This result is proved for the quantum Lorentz group by Podles and Woronowicz [27, Theorem 1.3]. Takeuchi [30] defines the quantum Lorentz group as a subalgebra of $\mathbb{C}_q[SL(2)] \otimes U_q(\mathfrak{sl}(2))^{\mathrm{cop}}$ and then goes on to prove that this subalgebra is isomorphic to $\mathbb{C}_q[D(SL(2))]$. A discussion of this map in a fairly general context appears in [22]. Although ξ is never surjective, it is in a certain sense not far away from being surjective. We now show that the image has a localization which is the invariant subring for the action of a certain finite group, Γ. This is only to be expected if we view this map as the quantization of the map $G \times G_r \to G \times G$ described in [17]. This map is a finite morphism onto the open subset GG_r of $G \times G$ whose fibre at every point is Γ.

Let $\Gamma = \{h \in H \mid h^2 = e\}$. We may identify the dual group $\hat{\Gamma}$ with the quotient $P/2P$.

LEMMA 3.7. $\mathbb{C}_q[D(G)]/(\operatorname{Ker} m + \operatorname{Ker} \theta) \cong \mathbb{C}[\tilde{\Gamma}]$.

Proof. Denote by $\mathbb{C}[\Gamma]$ the group algebra of Γ. Then there is a surjective Hopf algebra map $\eta\colon \mathbb{C}_q[D(G)] \to \mathbb{C}[\tilde{\Gamma}] = \mathbb{C}[\Gamma]^*$ given by

$$\eta\left(c^{\lambda}_{-\mu,\lambda} \otimes c^{\lambda'}_{-\mu',\lambda'}\right)(h) = \lambda(h)\lambda(h')\epsilon\left(c^{\lambda}_{-\mu,\lambda} c^{\lambda'}_{-\mu',\lambda'}\right).$$

It is easily checked that $\operatorname{Ker}\eta \supset \operatorname{Ker}\theta + \operatorname{Ker} m$. Conversely, let t_λ be the image of $c^{\lambda}_{-\lambda,\lambda} \otimes 1$ in $\mathbb{C}_q[D(G)]/(\operatorname{Ker} m + \operatorname{Ker}\theta)$. Then $t_\lambda^2 = 1$, $t_\lambda t_\mu = t_{\lambda+\mu}$, and the t_λ span $\mathbb{C}_q[D(G)]/(\operatorname{Ker} m + \operatorname{Ker}\theta)$. Thus by comparing dimensions we obtain that $\operatorname{Ker}\eta = \operatorname{Ker}\theta + \operatorname{Ker} m$. ∎

Denote by $R(A)$ the group of one-dimensional representation of a Hopf algebra A. The group Γ embeds in $R(\mathbb{C}_q[G])$, $R(U_q(\mathfrak{g})^{\mathrm{cop}})$, and $R(\mathbb{C}_q[D(G)])$ in such a way that the embeddings commute with the induced maps $R(U_q(\mathfrak{g})^{\mathrm{cop}}) \to R(\mathbb{C}_q[D(G)])$ and $R(\mathbb{C}_q[G]) \to R(\mathbb{C}_q[D(G)])$. Recall that, for any Hopf algebra A, there are left and right translation maps $l, r\colon R(A) \to \operatorname{Aut}(A)$ given by

$$l_h(x) = \sum h^{-1}(x_1)x_2, \qquad r_h(x) = \sum x_1 h(x_2).$$

The left and right translation actions of Γ on $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})^{\mathrm{cop}}$ factor through left and right translation actions on $\mathbb{C}_q[D(G)]$. Define an action $\tilde{d}\colon \Gamma \to \operatorname{Aut}\mathbb{C}_q[D(G)] \otimes \mathbb{C}_q[D(G)]$ by

$$\tilde{d}_h(x \otimes y) = r_h(x) \otimes l_h(y).$$

Consider Γ acting similarly on $(\mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}})$.

LEMMA 3.8. *The image of the comultiplication* $\Delta\colon \mathbb{C}_q[D(G)] \to \mathbb{C}_q[D(G)] \otimes \mathbb{C}_q[D(G)]$ *is contained in the subring of invariants,* $(\mathbb{C}_q[D(G)] \otimes \mathbb{C}_q[D(G)])^{\Gamma}$. *Hence the image of* $\mathbb{C}_q[D(G)]$ *under* ξ *is contained in the invariants* $(\mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}})^{\Gamma}$.

Proof. Notice that for $c_{\mu,\lambda} \in \mathbb{C}_q[G]$ and $h \in \Gamma$,

$$l_h(c_{\mu,\lambda}) = \mu(h)c_{\mu,\lambda}, \qquad r_h(c_{\mu,\lambda}) = \lambda(h)c_{\mu,\lambda}.$$

Hence, for $c_{\mu,\lambda} \otimes c_{\mu',\lambda'} \in \mathbb{C}_q[D(G)]$,

$$\begin{aligned}\tilde{d}_h\big(\Delta(c_{\mu,\lambda} \otimes c_{\mu',\lambda'})\big) &= \sum r_h\big(c_{\mu,\lambda_i} \otimes c_{\mu',\lambda'_j}\big) \otimes l_h\big(c_{-\lambda_i,\lambda} \otimes c_{-\lambda'_j,\lambda'}\big) \\ &= \sum \lambda_i(h)\lambda'_j(h)(-\lambda_i)(h)(-\lambda'_j)(h) c_{\mu,\lambda_i} \\ &\qquad \otimes c_{\mu',\lambda'_j} \otimes c_{-\lambda_i,\lambda} \otimes c_{-\lambda'_j,\lambda'} \\ &= \Delta\big(c_{\mu,\lambda} \otimes c_{\mu',\lambda'}\big).\end{aligned}$$

The second assertion follows from the first because $\theta \otimes m$ is a Γ-equivalent map. ∎

THEOREM 3.9. *The set* $\mathcal{S} = \{l \otimes k_{-\lambda} \mid \lambda \in 2\mathbf{L}^+\}$ *is an Ore subset of* $\xi(\mathbb{C}_q[D(G)])$. *The map* ξ *extends to an algebra isomorphism*

$$\xi\colon \mathbb{C}_q[D(G)]_{\mathcal{S}} \to \big(\mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}}\big)^{\Gamma}.$$

Proof. We first show that $\mathcal{S} \subset \xi(\mathbb{C}_q[D(G)])$. Notice that, for $\lambda \in \mathbf{L}^+$,

$$\xi\big(c^{\lambda}_{\nu,\lambda} \otimes 1\big) = \sum c^{\lambda}_{\nu,\mu_i} \otimes l^+\big(c^{\lambda}_{-\mu_i,\lambda}\big) = c^{\lambda}_{\nu,\lambda} \otimes k_{-\lambda}.$$

Similarly,

$$\xi\big(1 \otimes c^{\lambda}_{\nu,w_0\lambda}\big) = c^{\lambda}_{\nu,w_0\lambda} \otimes k_{w_0\lambda}.$$

Hence if

$$\Delta\big(c^{\lambda}_{-\lambda,\lambda}\big) = c^{\lambda}_{-\lambda,\mu_i} \otimes c^{\lambda}_{-\mu_i,\lambda}$$

then

$$1 = \epsilon\big(c^{\lambda}_{-\lambda,\lambda}\big) = \sum S\big(c^{\lambda}_{-\lambda,\mu_i}\big) c^{\lambda}_{-\mu_i,\lambda}.$$

Hence

$$\sum \xi\big(1 \otimes c^{-w_0\lambda}_{\mu_i,-\lambda}\big)\xi\big(c^{\lambda}_{-\mu_i,\lambda} \otimes 1\big) = \sum \big(c^{-w_0\lambda}_{\mu_i,-\lambda} c^{\lambda}_{-\mu_i,\lambda}\big) \otimes k_{-2\lambda} = 1 \otimes k_{-2\lambda}$$

as required.

To prove the result it now remains to show that

$$\forall r \in \big(\mathbb{C}_q[G] \otimes U_q(\mathfrak{g})^{\mathrm{cop}}\big)^{\Gamma}, \qquad \exists s \in \mathcal{S} \text{ such that } rs \in \xi\big(\mathbb{C}_q[D(G)]\big). \tag{3.2}$$

Notice that, for $\lambda \in \mathbf{L}$ and $h \in \Gamma$,

$$d_h(1 \otimes k_\lambda) = \lambda(h) 1 \otimes k_\lambda.$$

From this it follows that

$$\left(\mathbb{C}_q[G]\otimes U_q(\mathfrak{g})^{\mathrm{cop}}\right)^\Gamma=\left(\mathbb{C}_q[G]\otimes U^0\right)^\Gamma\left(1\otimes U_q(\mathfrak{g})^{\mathrm{cop}}\right)^\Gamma.$$

Since S is contained in the units of $U_q(\mathfrak{g})$, it suffices to verify condition (3.2) for the two factors separately.

By [19, 9.2.2], $\mathbb{C}_q[G]\otimes U^0$ is spanned by elements of the form

$$c^\lambda_{\nu,w_0\lambda}c^{\lambda'}_{\nu',\lambda'}\otimes k_\mu,$$

where $\nu,\nu',\mu\in\mathbf{L}$ and $\lambda,\lambda'\in\mathbf{L}^+$. This element is invariant if and only if $w_0\lambda+\lambda'+\mu\in 2\mathbf{L}$. On the other hand the image of $\mathbb{C}_q[D(G)]$ contains

$$\xi\left(1\otimes c^\lambda_{\nu,w_0\lambda}\right)\xi\left(c^{\lambda'}_{\nu',\lambda'}\otimes 1\right)=c^\lambda_{\nu,w_0\lambda}c^{\lambda'}_{\nu',\lambda'}\otimes k_{w_0\lambda-\lambda'}.$$

Now, for any $\gamma\in\mathbf{L}$, there exists a $\gamma'\in\mathbf{L}^+$ such that $\gamma'-\gamma\in\mathbf{L}^+$. Thus if $w_0\lambda+\lambda'+\mu=2\gamma$, then

$$\mu-2(\gamma'-w_0\lambda)=w_0\lambda-\lambda'-2(\gamma'-\gamma).$$

Hence,

$$\left(c^\lambda_{\nu,w_0\lambda}c^{\lambda'}_{\nu',\lambda'}\otimes k_\mu\right)\left(1\otimes k_{-2(\gamma'-w_0\lambda)}\right)=\xi\left(c^\lambda_{\nu,w_0\lambda}\otimes c^{\lambda'}_{\nu',\lambda'}\right)\left(1\otimes k_{-2(\gamma'-\gamma)}\right),$$

which lies in $\xi(\mathbb{C}_q[D(G)])$.

Notice that $(U_q(\mathfrak{g})^{\mathrm{cop}})^\Gamma$ is generated over $(U^0)^\Gamma$ by the elements f_ik_i and e_i. Hence it remains to verify condition (3.2) for these elements. Let $\lambda\in\mathbf{L}^+$, let α_i be a simple root, set $\mu=\lambda-\alpha_i$, and let $\nu\in\mathbf{L}$. Then

$$\begin{aligned}\xi\left(c^\lambda_{\nu,\mu}\otimes 1\right)&=\sum c^\lambda_{\nu,\mu_j}\otimes l^+\left(c^\lambda_{-\mu_j,\mu}\right)\\&=c^\lambda_{\nu,\lambda}\otimes l^+\left(c^\lambda_{-\lambda,\mu}\right)+c^\lambda_{\nu,\mu}\otimes l^+\left(c^\lambda_{-\mu,\mu}\right).\end{aligned}$$

Now $l^+(c^\lambda_{-\lambda,\mu})=f_ik_{-\mu}$ and $c^\lambda_{\nu,\mu}\otimes l^+(c^\lambda_{-\mu,\mu})\in(\mathbb{C}_q[G]\otimes U^0)^\Gamma$. Hence by the paragraph above, $\xi(\mathbb{C}_q[D(G)])$ contains $c^\lambda_{\nu,\lambda}\otimes f_ik_{-\mu}k_\gamma$ for some $\gamma\in-2\mathbf{L}^+$. As noted above,

$$\xi\left(1\otimes c^{-w_0\lambda}_{\mu_j,-\lambda}\right)=c^{-w_0\lambda}_{\mu_j,-\lambda}\otimes k_{-\lambda}.$$

Hence, for a suitably chosen $\gamma\in\mathbf{L}^+$, $\xi(\mathbb{C}_q[D(G)])$ contains

$$\sum\left(c^{-w_0\lambda}_{\mu_j,-\lambda}\otimes k_{-\lambda}\right)\left(c^\lambda_{-\mu_j,\lambda}\otimes f_ik_{-\mu}k_\gamma\right)=(1\otimes f_ik_i)\left(1\otimes k_{\gamma-2\lambda}\right)$$

as required. A similar argument works for the element e_i. ∎

4. NOETHERIANITY

We now prove that the double of a standard quantum group $\mathbb{C}_q[G]$ is again noetherian. We use the approach of Brown and Goodearl [4]. Since both $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ are homomorphic images of this algebra, this provides a straightforward and unified proof that both $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ are noetherian. Recall that if β is a braiding on a Hopf algebra A, then the braiding on the category of right comodules is given by the maps $\beta_{V\otimes W}: V \otimes W \to W \otimes V$, where

$$\beta(v \otimes w) = \sum w_0 \otimes v_0 \beta(v_1, w_1)$$

(see, for instance, [15]).

DEFINITION. Let (A, β) be a braided Hopf algebra and let V be a finite-dimensional right A-comodule. A full flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of subspaces of V is said to be *β-invariant* if

$$\beta_{V\otimes V}(V_i \otimes V) = V \otimes V_i.$$

The flag is said to be *strongly β-invariant* if

$$\beta_{V\otimes W}(V_i \otimes W) = W \otimes V_i$$

for all right comodules W.

If V is a right A-comodule with basis v_i for $i \in I$ and structure map $\rho: V \to V \otimes A$, then the subspace of A spanned by

$$\left\{a_{ij} \mid \rho(v_j) = \sum v_i \otimes a_{ij}\right\}$$

is denoted by $C(V)$ [1, p. 129].

DEFINITION. A right comodule V over a Hopf algebra A is said to be a *generator* for A if $A = k\langle C(V)\rangle$, or, equivalently, if $A = \sum_j C(V^{\otimes j})$.

One of the main results of [4] is the following (slightly reworked into the language of braided Hopf algebras).

THEOREM 4.1. *Let (A, β) be a braided Hopf algebra. Suppose there exists a finite dimensional right A-comodule V such that*

1. *V is a generator for A;*
2. *V has a β-invariant flag.*

Then A is noetherian.

Proof. See [4, Theorem 4.4]. ∎

The key result is the following.

LEMMA 4.2. *All finite dimensional right* $\mathbb{C}_q[D(G)]$*-comodules have a strongly* γ'*-invariant flag*.

Proof. It suffices to prove the result for comodules of the form $V \otimes V'$ for comodules V and V' over $\mathbb{C}_q[G]$. It follows from the form of β given above that there exist strongly β-invariant flags of V and V' such that

$$\beta_{V \otimes V'}\left(V_i \otimes V_j'\right) = V_j' \otimes V_i.$$

Now let W and W' be two other such $\mathbb{C}_q[G]$-comodules. Notice that

$$\gamma'_{V \otimes V', W \otimes W'} = \beta_{14}(\tau\beta_{13})\left(\tau\beta_{42}^{-1}\right)\left(\tau\beta_{32}^{-1}\right),$$

where β_{kl} denotes the map induced from β on the kth and lth components. Using this one can observe easily that

$$\gamma'_{V \otimes V', W \otimes W'}\left(V_i \otimes V_j' \otimes W \otimes W'\right) = W \otimes W' \otimes V_i \otimes V_j'.$$

From this it follows easily that $V \otimes V'$ has a strongly γ'-invariant flag. ∎

THEOREM 4.3. *The algebra* $\mathbb{C}_q[D(G)]$ *is noetherian for any connected semisimple algebraic group* G.

Proof. It remains to notice that $\mathbb{C}_q[D(G)]$ has a finite dimensional right comodule generator. The argument is the same as in the classical case. ∎

COROLLARY 4.4. *The algebras* $\mathbb{C}_q[G]$ *and* $U_q(\mathfrak{g})$ *are noetherian.*

5. FURTHER NONSTANDARD QUANTUM GROUPS

We now discuss briefly a generalization of double quantum groups and a lifting theorem which provides a method of constructing new families of nonstandard quantum groups corresponding to certain families of solutions of the modified classical Yang–Baxter equation.

We begin again with a very general result on lifting of Hopf algebra twists. The proof is routine.

THEOREM 5.1. *Let* $\phi\colon A \to B$ *be a homomorphism of Hopf algebras and let* σ *be a* 2*-cocycle on* B. *Then* $\sigma' = \phi^*(\sigma)$ *is a* 2*-cocycle. Moreover the induced map*

$$\phi\colon A_{\sigma'} \to B_\sigma$$

between the corresponding twisted Hopf algebras is a Hopf algebra homomorphism.

In particular, if C is a braided Hopf algebra and ϕ: $A \to C \otimes C$ is a Hopf algebra map, then there exists a 2-cocycle σ' on A such that the map

$$\phi\colon A_{\sigma'} \to C \bowtie C$$

is a Hopf algebra homomorphism.

In order to apply this result to standard quantum groups, one needs Hopf algebra maps of the form $\mathbb{C}_q[G] \to \mathbb{C}_{q'}[G' \times G']$. For this it is not quite enough to find morphisms $G' \times G' \to G$. Let us say that a morphism of $G' \to G$ of connected, simply connected semisimple groups is *admissible* if the induced map on the Lie algebras arises from a Dynkin diagram embedding as defined in [19, 10.4.5]. Then Braverman has shown that there is a Hopf algebra surjection $\mathbb{C}_q[G] \to \mathbb{C}_{q'}[G']$ and moreover that all such maps arise in this way [4].

THEOREM 5.2. *Let G and G' be connected, simply connected semisimple algebraic groups. Let ψ: $G' \times G' \to G$ be an admissible embedding. Then there is a 2-cocycle σ' on $\mathbb{C}_q[G]$ and an associated nonstandard quantum group $\mathbb{C}_{q,\psi}[G] = \mathbb{C}_q[G]_{\sigma'}$ such that*

1. $\mathbb{C}_{q,\psi}[G]$ *is a braided Hopf algebra*;
2. *the category* Comod$-\mathbb{C}_{q,\psi}[G]$ *is equivalent as a braided rigid monoidal category to* Comod$-\mathbb{C}_q[G]$.

Moreover there is a natural surjective homomorphism $\mathbb{C}_{q,\psi}[G] \to \mathbb{C}_{q'}[D(G')]$, where $q' = q^r$ for some rational number r.

For instance, if $2m \leq n$, then we have admissible embeddings ψ: $SL(m,\mathbb{C}) \times SL(m,\mathbb{C}) \to SL(n,\mathbb{C})$. This yields some nonstandard quantum groups of the form $\mathbb{C}_{q,\psi}[SL(n)]$. Since the braiding γ is essentially the universal T-matrix, these quantizations appear to be related to some of the "esoteric quantum groups" constructed by Fronsdal and Galindo [13].

The connection between these quantum groups and the solutions of the modified classical Yang–Baxter equation classified by Belavin and Drinfeld appears to be the following. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G', respectively. Let $\mathbf{B}$ and $\mathbf{B}'$ be bases for the roots of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Then we may choose $\mathbf{B}$ and $\mathbf{B}'$ such that the map ψ induces an embedding of the Dynkin diagram $\mathbf{B}' \times \mathbf{B}'$ into $\mathbf{B}$. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the images of the two copies of $\mathbf{B}'$ in $\mathbf{B}$ and let τ: $\mathbf{B}_1 \to \mathbf{B}_2$ be the natural isomorphism. Then τ is a triple in the sense of [2]. We conjecture that $\mathbf{C}_{q,\psi}[G]$ can be regarded as a deformation in the algebraic sense [8] of the algebra of functions on the group G with Poisson structure given by a solution of the modified Yang–Baxter equation associated with τ. On the other hand, given any triple $\tau\colon \mathbf{B}_1 \to \mathbf{B}_2$ in the sense of [2] which is disjoint in the sense that $(\alpha, \alpha') = 0$ for all $\alpha \in \mathbf{B}_1$ and $\alpha' \in \mathbf{B}_2$, then we may construct an admissible map of the form given in the theorem. Thus we may think of the quantum group as being constructed directly from this data.

Notice that these cocycle twists are deceptively simple. Although the coalgebra structure is preserved by the twist, the algebra structure is altered quite dramatically. Composing the homomorphism $\mathbb{C}_{q,\psi}[G] \to \mathbb{C}[D(G')]$ with the map $\mathbf{C}_{q'}[D(G')] \to U_{q'}(\mathfrak{g}')$ yields a Hopf algebra map $\mathbb{C}_{q,\psi}[G] \to U_{q'}(\mathfrak{g}')$. Thus $\mathbb{C}_{q,\psi}[G]$ has a category of finite dimensional representations equivalent to those of $U_{q'}(\mathfrak{g}')$. This is quite different from the case of the standard quantum groups where all finite dimensional representations are one-dimensional [19, 9.3.11]. On the other hand the existence of this map is consistent with the philosophy on nonstandard quantum groups advanced in [16]. Briefly the conjecture suggested by this work is the following. Let $\mathbb{C}_q[G, r]$ be an algebraic deformation of the group G with Poisson structure given by a solution r of the modified classical Yang–Baxter equation associated with a triple $(\tau, \mathbf{B}_1, \mathbf{B}_2)$. Then there should be a Hopf algebra homomorphism $\mathbb{C}_q[G, r] \to U_{q'}(\tilde{\mathfrak{g}}, \tilde{r})$, where $U_{q'}(\tilde{\mathfrak{g}}, \tilde{r})$ is the FRT-dual of the quantization $\mathbb{C}_{q'}[(\tilde{G}, \tilde{r})]$ of the reductive group $\tilde{G}$ associated with the reductive Lie algebra $\tilde{\mathfrak{g}}$ constructed in [16, Theorem 6.4]. The semisimple part of $\tilde{\mathfrak{g}}$ is the semisimple Lie algebra given by $\mathbf{B}_1$.

Using Theorem 5.1 we may easily generalize the notion of the double $A \bowtie A$ of a braided Hopf algebra A to a twisted product $A^{\bowtie n}$ of n copies of A in the following way. Using the map $m \otimes 1\colon (A \bowtie A) \otimes A \to A \otimes A$ and Theorem 5.1, we may twist $(A \bowtie A) \otimes A$ to obtain a new Hopf algebra $A^{\bowtie 3}$ which itself maps onto A. Continuing in this way, we may iteratively construct $A^{\bowtie (n+1)}$ from $A^{\bowtie n}$ using the map $A^{\bowtie n} \otimes A \to A \otimes A$ and Theorem 5.1. Clearly we may also view the construction $A^{\bowtie n}$ as a series of cocycle twists of the n-fold tensor product $A^{\otimes n} = A \otimes A \otimes A \otimes \cdots \otimes A$. Hence there exists a single cocycle τ such that $A^{\bowtie n} \cong A_\tau^{\otimes n}$.

Again we may lift this construction to obtain interesting families of nonstandard quantum groups. The algebra $\mathbb{C}_q[G]^{\bowtie n}$ can be thought of as a nonstandard quantization of $G \times G \times \cdots \times G$. If G and G' are simply connected, then any admissible embedding $G' \times G' \times \cdots \times G' \to G$ induces a homomorphism $\mathbb{C}_q[G] \to C_{q'}[G' \times \cdots \times G']$ and we may lift the twisting of $\mathbb{C}_q[G']^{\bowtie n}$ to a twisting on $\mathbb{C}_q[G]$ using Theorem 5.1. Thus, for instance, the natural block-diagonal embedding $\psi\colon SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ into $SL(6, \mathbb{C})$ yields a new quantum group $\mathbb{C}_{q,\psi}[SL(6)]$.

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