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An Example of a Primitive Polynomial Ring

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In [5, p. 201] Resco poses the following question: If the polynomial ring $R[X]$ is primitive, must R be primitive? We answer the question in the negative by exhibiting local rings (i.e., $R/\operatorname{Jac} R$ is a division ring) with arbitrarily large finite Krull dimensions (in the sense of Gabriel and Rentschler, see [2]) such that $R[X_1, ..., X_s]$ is primitive for all positive integers s.

Let D be a division ring with center C. Recall the definition of the transcendence degree of D given in [3]:

$$\operatorname{tr} \deg_C D = \sup\{\operatorname{tr} \deg_C F : F \text{ a subfield of } D\}.$$

Amitsur and Small show that if A_n is the nth Weyl algebra, its quotient division ring D_n has transcendence degree n [1]. Since $D_{n-1} \subseteq D_n$ we may take the union of these D_n and $\bigcup_n D_n$ is a division ring of infinite transcendence degree. Hence there exist division rings of all possible transcendence degrees.

Now let $T = D[Y_1, ..., Y_n]$ and let $P = (Y_1, ..., Y_n)$, the maximal ideal generated by $Y_1, ..., Y_n$. Let $S = T \backslash P$. By [6,2.2], S is an Ore set. Define $R = TS^{-1} = D[Y_1, ..., Y_n]_{(Y_1, ..., Y_n)}$. Then R has Jacobson radical PR, and $R/PR \cong D$ so that R is a local ring. By [2,9.4], $K \dim R \leqslant K \dim T = n$. On the other hand R contains chains of primes of length n, so $K \dim R$ must equal n [2, 7.9].

THEOREM. *Let D be a division ring with center C and let d be a positive integer such that* $\operatorname{tr} \deg_C D \geqslant d$. *Let $R = D[Y_1, ..., Y_n]_{(Y_1, ..., Y_n)}$ where $n \leqslant d$. Then for any positive integer $s \leqslant d - n + 1$, $R[X_1, ..., X_s]$ is primitive.*

This result follows immediately using a result of Resco's [4, 3.2] from the following proposition. Recall that for a Noetherian domain, the extended center is the center of the quotient division ring. For the rings considered here this will just be the quotient field of the center of the ring.

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Proposition. *Let C, D, R be as in the theorem. Then there exists a simple faithful $R[X]$ module N such that* $\operatorname{tr}\deg_F(\operatorname{End}_{R[X]}N) \geqslant d-n$, *where F is the extended center of $R[X]$.*

Proof. Pick a purely transcendental subfield $C(t_1,..., t_d)$ of D of transcendence degree d over C. Let $T=D[Y_1,..., Y_n]$, and make the Laurent polynomial ring $D[\zeta, \zeta^{-1}]$ into a right T-module via the action:

$$\begin{aligned} \text{for } f\in D[\zeta,\zeta^{-1}], \qquad & f\cdot Y_i = t_i f\zeta && \text{for } i=1,..., n-1 \\ & f\cdot Y_n = f\zeta && \\ & f\cdot d = fd && \text{for all } d\in D. \end{aligned}$$

Denote this module $M=M_T$.

For any $f\in M$, $\operatorname{ann}_T f$ is a homogeneous right ideal of T (i.e., generated by sums of monomials of the same degree). Hence if S is the Ore set $T\backslash(Y_1,..., Y_n)$, then $\operatorname{ann}_T f\cap S=\varnothing$, since elements of S have non-zero constant term. Thus, the natural map from M into the localized module $N=M\otimes_T TS^{-1}=M\otimes_T R$ is an embedding. Further, any element of M can be written in the form $\zeta^i g(\zeta)=\zeta^i\cdot g(Y_n)$ where $g(Y_n)$ has non-zero constant term; thus, N has a unique chain of submodules

$$N=M\otimes R\supseteq\cdots\supseteq\zeta^i\otimes R\supseteq\zeta^{i+1}\otimes R\supseteq\cdots\supseteq 0.$$

Now let λ: $C(t_1,..., t_d)\ [\zeta^{\pm 1}]\to\operatorname{End}_T M$ be the canonical embedding mapping elements of $C(t_1,..., t_d)\ [\zeta^{\pm 1}]$ to the endormorphism given by left multiplication. We will identify $C(t_1,..., t_d)\ [\zeta^{\pm 1}]$ with its canonical image under λ. Since R is an Ore localization of T, $\operatorname{End}_T(M)$ embeds in $\operatorname{End}_R(M\otimes R)$. Let X act on N via the canonical image of $t_n\zeta^{-1}$ so that $(\zeta^i\otimes r)X=t_n\zeta^{i-1}\otimes r$. Clearly this action makes N into a simple $R[X]$-module.

Non-zero two-sided ideals of $D[Y_1,..., Y_n, X]$ must intersect $C[Y_1,..., Y_n, X]$ non-trivially since D is a central simple C-algebra and $D[Y_1,..., Y_n, X]\cong D\otimes_C C[Y_1,..., Y_n, X]$. Since $R[X]$ is a localization of $D[Y_1,..., Y_n, X]$, the same must be true of non-zero ideals of $R[X]$. Hence, in particular, if N is not faithful, $\operatorname{ann}_{R[X]}N\cap C[Y_1,..., Y_n, X]\neq 0$. But the action of $C[Y_1,..., Y_n, X]$ on N induces a map ν: $C[Y_1,..., Y_n, X]\to\operatorname{End}_{R[X]}N$ whose image is $C[t_1\zeta,..., t_{n-1}\zeta, \zeta, t_n\zeta^{-1}]$. The transcendence degree of the latter is clearly $n+1$, so $\ker\nu=0$; thus $\operatorname{ann}_{R[X]}N\cap C[Y_1,..., Y_n, X]=0$ and N is faithful.

As remarked above, the extended center of $R[X]$ is just the center of its quotient division ring, namely, $C(Y_1,..., Y_n, X)$, and the canonical image of this field in $\operatorname{End}_{R[X]}N$ is $F=C(t_1,..., t_n, \zeta)$. But $F_1=C(t_1,..., t_d, \zeta)\subseteq$

$\mathrm{End}_{R[X]}N$ and $\mathrm{tr\,deg}_F F_1 = d - n$; thus $\mathrm{tr\,deg}_F \mathrm{End}_{R[X]}N \geqslant d - n$. This proves the proposition.

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