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Nonstandard Solutions of the Yang–Baxter Equation

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Abstract. Explicit solutions of the quantum Yang–Baxter equation are given corresponding to the non-unitary solutions of the classical Yang–Baxter equation for $\mathfrak{sl}(5)$.

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1. Introduction

Etingof and Kazhdan recently proved that any finite-dimensional Lie bialgebra $\mathfrak{g}$ may be quantized [3]. That is, there exists a topological Hopf algebra structure on $U(\mathfrak{g})[[h]]$ such that the Lie bialgebra structure on $\mathfrak{g}$ is the one induced on $\mathfrak{g}$ by passing to the 'semi-classical limit'. From this they deduced a general procedure for quantizing solutions of the classical Yang–Baxter equation (CYBE). Thus, at least in theory, one can construct solutions of the quantum Yang–Baxter equation from given solutions of the classical Yang–Baxter equation. Unfortunately, their procedure is not easy to implement explicitly, even in small dimensional situations.

In this note we exhibit an explicit answer to this problem for a particularly interesting family of Lie bialgebra structures on $\mathfrak{sl}(5)$. These are the bialgebra structures associated to nonunitary solutions of the CYBE (or equivalently of the modified classical Yang–Baxter equation (MCYBE)) as classified by Belavin and Drinfeld [1]. For each such solution of the CYBE we construct an R-matrix using the Gerstenhaber–Giaquinto–Schack (GGS) conjecture [4]. The YBE was verified in each case using *Mathematica.*

The GGS conjecture concerns the form of the quantization of such solutions of the CYBE in the case of $\mathfrak{sl}(n)$. The case of $\mathfrak{sl}(5)$ is to some extent the first interesting case. For $\mathfrak{sl}(2)$ there are no solutions of the MCYBE except the standard one. For $\mathfrak{sl}(3)$ the only non-standard solution is that associated to the well-known Cremmer–Gervais quantization and for $\mathfrak{sl}(4)$ the nonstandard solutions are essentially of three types, the Cremmer–Gervais solution and two other fairly simple examples. The corresponding R-matrices for the latter two types can be constructed using other techniques [6]. On the other hand for $\mathfrak{sl}(5)$ there are 13 different types of solutions to the MCYBE and for many of these the corresponding R-matrix was hitherto unknown. The validity of the GGS conjecture for $\mathfrak{sl}(5)$ gives strong evidence that the conjecture should be true for all n.

2. Solutions to the CYBE and Quantization

2.1. THE BELAVIN–DRINFELD DESCRIPTION OF SOLUTIONS TO THE CYBE

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Let Δ be the associated root system and Γ a set of simple roots. A classical r-matrix over $\mathfrak{g}$ is a an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Take an invariant bilinear from on form on $\mathfrak{g}$ and let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the associated Casimir element. In [1], Belavin and Drinfeld gave the following description of solutions of the CYBE which satisfy $r_{12} + r_{21} = t$. These are the 'nonunitary' solutions.

Let Γ_1, Γ_2 be two subsets of Γ and let $\tau\colon \Gamma_1 \to \Gamma_2$ be a bijection satisfying

(1) $(\tau\alpha, \tau\beta) = (\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$;
(2) For every $\alpha \in \Gamma_1$, there is a $k \geqslant 0$ with $\tau^k\alpha \in \Gamma_1$ but $\tau^{k+1}\alpha \notin \Gamma_1$.

The data $(\tau, \Gamma_1, \Gamma_2)$ (or more concisely just τ) is often called a *Belavin–Drinfeld triple*. Given such a triple τ, an element $r^0 \in \mathfrak{h} \otimes \mathfrak{h}$ is called τ-admissible if

(1) $r^0_{12} + r^0_{21} = t^0$,
(2) $(\tau\alpha \otimes 1)r^0 + (1 \otimes \alpha)r^0 = t^0$,

where t^0 is the component of t in $\mathfrak{h} \otimes \mathfrak{h}$. A τ-admissible r^0 is necessarily of the form $t^0/2 + \tilde{r}^0$, where $\tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h}$. The set of all $\tilde{r}^0$ forms a linear subvariety of $\mathfrak{h} \wedge \mathfrak{h}$ of dimension $\binom{d}{2}$ where $d = \#(\Gamma - \Gamma_1)$.

Now τ can be extended to an isomorphism of Lie subalgebras $\tau\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_i$ is the Lie subalgebra of $\mathfrak{g}$ associated to Γ_i. Choose $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = 1$ and $\tau(e_\alpha) = e_{\tau\alpha}$ and define an ordering on Δ by $\alpha \prec \beta$ if $\tau^k\alpha = \beta$ for some positive integer k. View $\mathfrak{g} \wedge \mathfrak{g}$ as a subset of $\mathfrak{g} \otimes \mathfrak{g}$ via the identification $x \wedge y = 1/2(x \otimes y - y \otimes x)$. Then Belavin and Drinfeld showed [1] that

$$r = r^0 + \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha + \sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha} \wedge e_\beta$$

is a solution of the Yang–Baxter equation satisfying $r_{12} + r_{21} = t$ and that every such solution is of this form for some choice of $\mathfrak{h}$, Γ, τ and r^0.

For any $\mathfrak{g}$ there is the 'trivial' triple which has $\Gamma_1 = \Gamma_2 = \emptyset$ and $\tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h}$ arbitrary. A particularly interesting triple for $\mathfrak{sl}(n)$ is the 'Cremmer–Gervais' triple which has

$$\Gamma_1 = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}, \qquad \Gamma_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}\}, \quad \text{and} \quad \tau(\alpha_i) = \alpha_{i-1}.$$

In contrast to the trivial triple, there is a unique admissible r^0 for the Cremmer–Gervais triple.

2.2. THE GERSTENHABER–GIAQUINTO–SCHACK CONJECTURE

The Gerstenhaber–Giaquinto–Schack conjecture is a conjectured form for the quantization of the above classical r-matrices in the case where $\mathfrak{g} = \mathfrak{sl}(n)$, considered as a subset of $M_n(\mathbb{C})$. In this setting, a quantization of a classical r-matrix is an $R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ which has semi-classical limit r and satisfies the quantum Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Take the form to be the trace form $(x, y) = \mathrm{Tr}(xy)$ and let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices of trace zero. The standard Cartan–Weyl basis is then

$$e_{\alpha_i} = e_{i,i+1}, \qquad e_{-\alpha_i} = e_{i+1,i} \quad \text{and} \quad h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}] = e_{ii} - e_{i+1,i+1}.$$

Let τ be a Belavin–Drinfeld triple as described above and let $r^0 \in \mathfrak{h} \otimes \mathfrak{h}$ be τ-admissible. Set

$$a = \sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha} \wedge e_{\beta}, \qquad c_+ = \sum_{\alpha>0} e_{-\alpha} \otimes e_{\alpha}, \quad \text{and} \quad c = \sum_{\alpha>0} e_{-\alpha} \wedge e_{\alpha}.$$

Set $\epsilon = -(ac + ca + a^2)$. Now define $\tilde{a}$ by

$$\tilde{a} = \sum a_{jl}^{ik} q^{a_{jl}^{ik}\epsilon_{jl}^{ik}} e_{ij} \otimes e_{kl},$$

where $a = \sum a_{jl}^{ik} e_{ij} \otimes e_{kl}$ and similarly for ϵ. Set $\tilde{q} = q - q^{-1}$. The standard R-matrix is then

$$R_s = q^{t^0 + 1/n} + \tilde{q}c_+ = q\sum_i e_{ii} \otimes e_{ii} + \sum_{i\neq j} e_{ii} \otimes e_{jj} + \tilde{q}\sum_{i>j} e_{ij} \otimes e_{ji}.$$

It is easy to check that R_s satisfies the quantum Yang–Baxter equation and that PR_s satisfies the Hecke relation $(PR_s - q)(PR_s + q^{-1}) = 0$ where P is the permutation matrix.

GERSTENHABER–GIAQUINTO–SCHACK CONJECTURE. *Let τ be a Belavin–Drinfeld triple for $\mathfrak{sl}(n)$ and suppose $r^0 = t^0/2 + \tilde{r}^0$ is τ-admissible. Then the* *matrix* $R = q^{\tilde{r}^0}(R_s + \hat{q}\tilde{a})q^{\tilde{r}^0}$ *satisfies the quantum Yang–Baxter equation and* PR *satisfies the Hecke relation.*

Taking τ to be the trivial triple yields the standard R-matrix when $\tilde{r}^0 = t^0/2$ and the standard multiparameter R-matrices when $\tilde{r}^0$ is arbitrary. For use later, let $R(r^0) = q^{\tilde{r}^0}(R_s)q^{\tilde{r}^0}$ denote the standard multiparameter R-matrix. As is well known, if $r^0 = t^0/2 + \sum_{i<j} c_{ij} e_{ii} \wedge e_{jj}$ then

$$R(r^0) = q\sum_{i} e_{ii} \otimes e_{ii} + \sum_{i<j}(q^{c_{ij}} e_{ii} \otimes e_{jj} + q^{-c_{ij}} e_{jj} \otimes e_{ii}) + \hat{q}\sum_{i>j} e_{ij} \otimes e_{ji}.$$

For the Cremmer–Gervais triples described above the formula gives the Cremmer–Gervais R-matrices [2].

2.3. THE GGS CONJECTURE FOR $\mathfrak{sl}(5)$

We now consider the explicit form of the R-matrices associated to the Belavin–Drinfeld triples on $\mathfrak{sl}(5)$. According to the GGS Conjecture, each R is of the form $R(r^0) + \hat{q}q^{\tilde{r}^0}\tilde{a}q^{\tilde{r}^0}$ for an admissible r^0. The specific form of $R(r^0)$ has already been exhibited. The other summand, $q^{\tilde{r}^0}\tilde{a}q^{\tilde{r}^0}$, is always a sum of 'quantized' wedge products. Specifically, for positive roots α and β and any constant c, set $e_{-\alpha} \wedge_c e_\beta = q^{-c} e_{-\alpha} \otimes e_\beta - q^c e_\beta \otimes e_{-\alpha}$. For all triples, the term $q^{\tilde{r}^0}\tilde{a}q^{\tilde{r}^0}$ is always of the form

$$\sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha} \wedge_{c(\alpha,\beta)} e_\beta,$$

where the constants $c(\alpha, \beta)$ are determined by $\tilde{r}^0$ and ϵ.

Denote by $\mathcal{T}$ the set of triples on $\mathfrak{sl}(5)$. Notice that if $(\tau, \Gamma_1, \Gamma_2)$ is a triple, then $(\tau^{-1}, \Gamma_2, \Gamma_1)$ is also a triple. Also the graph automorphism of A_4 induces a bijection on the set of triples. Since these two involutions of $\mathcal{T}$ commute, this gives an action of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{T}$.

PROPOSITION 2.1. *The Gerstenhaber–Giaquinto–Schack conjecture is true for* $n = 5$. *The triples below comprise a complete set of representatives from the* 13 *orbits under the action of* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *on* $\mathcal{T}$. *For each triple the generic admissible* r^0 *and the Hecke* R*-matrix produced by the GGS conjecture are also explicitly given.*

(1) $|\Gamma_1| = 3$

(a) *The 'Cremmer–Gervais' triple:* $\Gamma_1 = \{\alpha_2, \alpha_3, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_2, \alpha_3\}$, $\tau(\alpha_i) = \alpha_{i-1}$*:*

$$\begin{aligned} r^0 \;=\; & t^0/2 + \tfrac{1}{5}(-3h_{\alpha_1} \wedge h_{\alpha_2} - 4h_{\alpha_1} \wedge h_{\alpha_3} - 3h_{\alpha_1} \wedge h_{\alpha_4} - \\ & -4h_{\alpha_2} \wedge h_{\alpha_3} - 4h_{\alpha_2} \wedge h_{\alpha_4} - 3h_{\alpha_3} \wedge h_{\alpha_4}), \end{aligned}$$

$$\begin{aligned} R &= R(r^0) + \hat{q}(e_{54} \wedge_{2/5} e_{34} + e_{54} \wedge_{4/5} e_{23} + \\ &\quad + e_{54} \wedge_{6/5} e_{12} + e_{43} \wedge_{2/5} e_{23} + \\ &\quad + e_{43} \wedge_{4/5} e_{12} + e_{32} \wedge_{2/5} e_{12} + e_{53} \wedge_{2/5} e_{24} + e_{53} \wedge_{4/5} e_{13} + \\ &\quad + e_{42} \wedge_{2/5} e_{13} + e_{52} \wedge_{2/5} e_{14}). \end{aligned}$$

(b) *The 'generalized Cremmer–Gervais' triple:* $\Gamma_1 = \{\alpha_1, \alpha_3, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_2, \alpha_4\}$, $\tau(\alpha_i) = \alpha_j$, *where* $j \equiv i + 3 \pmod 5$:

$$\begin{aligned} r^0 &= t^0/2 + \tfrac{1}{5}(h_{\alpha_1} \wedge h_{\alpha_2} - 2h_{\alpha_1} \wedge h_{\alpha_3} + h_{\alpha_1} \wedge h_{\alpha_4} - \\ &\quad - 2\, h_{\alpha_2} \wedge h_{\alpha_3} - 2h_{\alpha_2} \wedge h_{\alpha_4} + h_{\alpha_3} \wedge h_{\alpha_4}), \\ R &= R(r^0) + \hat{q}(e_{54} \wedge_{2/5} e_{23} + e_{21} \wedge_{4/5} e_{23} + \\ &\quad + e_{43} \wedge_{6/5} e_{23} + e_{21} \wedge_{2/5} e_{45} + \\ &\quad + e_{43} \wedge_{4/5} e_{45} + e_{43} \wedge_{2/5} e_{12} + e_{53} \wedge_{2/5} e_{13}). \end{aligned}$$

(2) $|\Gamma_1| = 2$

(a) $\Gamma_1 = \{\alpha_3, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_2\}$, $\tau(\alpha_i) = \alpha_{i-2}$:

$$\begin{aligned} r^0 &= t^0/2 + ch_{\alpha_1} \wedge h_{\alpha_2} + ((c-1)/2)\, h_{\alpha_1} \wedge h_{\alpha_3} + ch_{\alpha_1} \wedge h_{\alpha_4} - \\ &\quad - ((1+3c)/4)\, h_{\alpha_2} \wedge h_{\alpha_3} + ((c-1)/2)\, h_{\alpha_2} \wedge h_{\alpha_4} + \\ &\quad + ch_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{43} \wedge_{(3+c)/8} e_{12} + e_{54} \wedge_{(3+c)/8} e_{23} + e_{53} \wedge_{(1-c)/2} e_{13}).$$

(b) $\Gamma_1 = \{\alpha_3, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_2\}$, $\tau(\alpha_i) = \alpha_{5-i}$:

$$\begin{aligned} r^0 &= t^0/2 + \tfrac{1}{5}(-ch_{\alpha_1} \wedge h_{\alpha_2} - (1+c)h_{\alpha_1} \wedge h_{\alpha_3} - 3h_{\alpha_1} \wedge h_{\alpha_4} - \\ &\quad - 2h_{\alpha_2} \wedge h_{\alpha_3} + (c-1)h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4}), \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{54} \wedge_{3/5} e_{12} + e_{43} \wedge_{4/5} e_{23} + e_{53} \wedge_{9/5} (-e_{13})).$$

(c) $\Gamma_1 = \{\alpha_2, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_3\}$, $\tau(\alpha_i) = \alpha_{i-1}$:

$$\begin{aligned} r^0 &= t^0/2 + ch_{\alpha_1} \wedge h_{\alpha_2} + (1+3c)h_{\alpha_1} \wedge h_{\alpha_3} + \\ &\quad + (8c/3 + 1)h_{\alpha_1} \wedge h_{\alpha_4} + (1+3c)h_{\alpha_2} \wedge h_{\alpha_3} + \\ &\quad + (1+3c)h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$R = R(r^0) + \hat{q}h(e_{32} \wedge_{1+c} e_{12} + e_{54} \wedge_{1+c} e_{34}).$$

(d) $\Gamma_1 = \{\alpha_2, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_3\}$, $\tau(\alpha_4) = \alpha_1$, $\tau(\alpha_2) = \alpha_3$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + \tfrac{1}{5}((2-c)h_{\alpha_1} \wedge h_{\alpha_2} + \\ &\qquad\qquad + (1-c)h_{\alpha_1} \wedge h_{\alpha_3} - 3h_{\alpha_1} \wedge h_{\alpha_4} + \\ &\qquad\qquad + 2h_{\alpha_2} \wedge h_{\alpha_3} + (c-1)\, h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4}), \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{54} \wedge_{2/5} e_{12} + e_{32} \wedge_{2/5} e_{34}).$$

(e) $\Gamma_1 = \{\alpha_1, \alpha_3\}$, $\Gamma_2 = \{\alpha_1, \alpha_4\}$, $\tau(\alpha_i) = \alpha_j$, *where* $j \equiv i + 3 \pmod 5$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + ((1-3c)/2)h_{\alpha_1} \wedge h_{\alpha_2} + \\ &\quad +((c-1)/2)h_{\alpha_1} \wedge h_{\alpha_3} + ch_{\alpha_1} \wedge h_{\alpha_4} + \\ &\quad +(3c-1)\, h_{\alpha_2} \wedge h_{\alpha_3} + (3c-1)h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{43} \wedge_{(1+3c)/4} e_{12} + e_{21} \wedge_{(1+3c)/4} e_{45} + e_{43} \wedge_{1-c} e_{45}).$$

(f) $\Gamma_1 = \{\alpha_3, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_2\}$, $\tau(\alpha_i) = \alpha_{i-1}$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + ((c-3)/6)h_{\alpha_1} \wedge h_{\alpha_2} + \\ &\quad +((c-1)/2)h_{\alpha_1} \wedge h_{\alpha_3} + ch_{\alpha_1} \wedge h_{\alpha_4} + \\ &\quad +((c-1)/2)\, h_{\alpha_2} \wedge h_{\alpha_3} + (4c/3)h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$\begin{aligned} R = R(r^0) + \hat{q}(&e_{32} \wedge_{1/2+c/6} e_{12} + e_{43} \wedge_{1/2+c/6} e_{23} + \\ &\qquad + e_{43} \wedge_{1+c/3} e_{12} + e_{42} \wedge_{1/2+c/6} e_{13}). \end{aligned}$$

(3) $|\Gamma_1| = 1$

(a) $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_2\}$, $\tau(\alpha_1) = \alpha_2$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + ((1+y)/3)h_{\alpha_1} \wedge h_{\alpha_2} + yh_{\alpha_1} \wedge h_{\alpha_3} + \\ &\quad +((3z-x)/3)h_{\alpha_1} \wedge h_{\alpha_4} + \\ &\quad +yh_{\alpha_2} \wedge h_{\alpha_3} + zh_{\alpha_2} \wedge h_{\alpha_4} + xh_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{21} \wedge_{(2-y)/3} e_{23}).$$

(b) $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_3\}$, $\tau(\alpha_1) = \alpha_3$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + ((z-2y)/2)h_{\alpha_1} \wedge h_{\alpha_2} + \\ &\quad +((1+x)/2)h_{\alpha_1} \wedge h_{\alpha_3} + xh_{\alpha_1} \wedge h_{\alpha_4} + \end{aligned}$$

$$+y\, h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4},$$

$$R = R(r^0) + \hat{q}(e_{21} \wedge_{(z-2)/4} e_{34}).$$

(c) $\Gamma_1 = \{\alpha_1\}$, $\Gamma_2 = \{\alpha_4\}$, $\tau(\alpha_1) = \alpha_4$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + ((y-2z)/2)h_{\alpha_1} \wedge h_{\alpha_2} + ((y-2x)/2)h_{\alpha_1} \wedge h_{\alpha_3} + \\ &\quad +((1-x+z)/2)h_{\alpha_1} \wedge h_{\alpha_4} + \\ &\quad +y h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{21} \wedge_{(y+2)/4} e_{45}).$$

(d) $\Gamma_1 = \{\alpha_2\}$, $\Gamma_2 = \{\alpha_3\}$, $\tau(\alpha_2) = \alpha_3$:

$$\begin{aligned} r^0 \;&=\; t^0/2 + (-1+3y-z)h_{\alpha_1} \wedge h_{\alpha_2} + \\ &\quad +(-1-x+3y)h_{\alpha_1} \wedge h_{\alpha_3} + 3(z-x)\, h_{\alpha_1} \wedge h_{\alpha_4} + \\ &\quad +y h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4}, \end{aligned}$$

$$R = R(r^0) + \hat{q}(e_{32} \wedge_{1-x-y+z} e_{34}).$$

(4) $|\Gamma_1| = 0$ *The 'trivial triple'* $\Gamma_1 = \Gamma_2 = \emptyset$:

$r^0 = t^0/2 + \tilde{r}^0$ *with* $\tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h}$ *arbitrary*,

$R = R(r^0)$ *is the standard multiparameter* R*-matrix.*

Perhaps the most interesting new R-matrix is that associated to type $1(b)$, the generalized Cremmer–Gervais triple. Like the Cremmer–Gervais triple, its Γ_1, which must omit at least one root, omits precisely one and thus its r^0 is uniquely determined. Setting $\hat{p} = -\hat{q}$, the matrix form of the generalized Cremmer–Gervais R-matrix is shown in Table I

3. Conclusion

We have constructed here quantizations of each type of non-unitary solution of the classical Yang–Baxter equation for $\mathfrak{sl}(5)$. In so doing we verified in this case the conjecture of Gerstenhaber, Giaquinto and Schack. This gives further evidence that the GGS conjecture should be true for all Belavin–Drinfeld triples on $\mathfrak{sl}(n)$.

One can proceed in the usual way to construct for each of these R, a quantization of $\mathbb{C}[\mathrm{SL}(5)]$, the algebra of algebraic functions on $\mathrm{SL}_{\mathbb{C}}(5)$. First one constructs the associated bialgebra $A(R)$. Using a case-by-case analysis one can see that the Poincare series of the associated quantum space and exterior algebra are the same as in the commutative case. Thus $A(R)$ contains a group-like q-determinant

Table I

$$\begin{pmatrix}
q&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&q^{\frac{1}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&q^{-\frac{3}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&q^{\frac{3}{5}}&0&0&0&\hat{p}q^{\frac{2}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&q^{-\frac{1}{5}}&0&0&0&0&0&0&0&\hat{p}q^{\frac{2}{5}}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\hat{q}&0&0&0&q^{-\frac{1}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\hat{q}q^{-\frac{4}{5}}&0&0&0&q&0&0&0&\hat{p}q^{\frac{4}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&q^{\frac{1}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\hat{q}q^{-\frac{2}{5}}&0&0&0&q^{-\frac{3}{5}}&0&0&0&\hat{p}q^{\frac{6}{5}}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&q^{\frac{3}{5}}&0&0&0&\hat{p}q^{\frac{2}{5}}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\hat{q}&0&0&0&0&0&0&0&q^{\frac{3}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\hat{q}&0&0&0&q^{-\frac{1}{5}}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&q&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&q^{\frac{1}{5}}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&q^{-\frac{3}{5}}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\hat{q}&0&0&0&0&0&0&0&\hat{q}q^{-\frac{2}{5}}&0&0&0&q^{-\frac{3}{5}}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\hat{q}&0&0&0&\hat{q}q^{-\frac{6}{5}}&0&0&0&q^{\frac{3}{5}}&0&0&0&\hat{p}q^{\frac{2}{5}}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&q^{-\frac{1}{5}}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}q^{-\frac{4}{5}}&0&0&0&q&0&0&0&\hat{p}q^{\frac{4}{5}}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q^{\frac{1}{5}}&0&0&0&0&0\\
0&0&0&0&\hat{q}&0&0&0&0&0&0&0&\hat{q}q^{-\frac{2}{5}}&0&0&0&0&0&0&0&q^{\frac{1}{5}}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&0&0&0&0&\hat{q}q^{-\frac{2}{5}}&0&0&0&q^{-\frac{3}{5}}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&0&0&0&0&q^{\frac{3}{5}}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&q^{-\frac{1}{5}}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q
\end{pmatrix}.$$

element D which turns out to be central. Hence one may define a Hopf algebra structure on $\mathbb{C}_R[\mathrm{SL}(5)] = A(R)/(D-1)$. Since R is a Hecke symmetry in the sense of Gurevich, it is possible to exploit some Hecke algebra techniques to show that the category of comodules over these Hopf algebras is equivalent as a rigid monoidal category to the category of comodules over $\mathbb{C}_q[\mathrm{SL}(5)]$ [5, 7, 8]. Hence these R-matrices do produce genuine nonstandard quantizations of $\mathbb{C}[\mathrm{SL}(5)]$.

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