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On Ore and Skew-Laurent Extensions of Noetherian Rings

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We investigate the relation between the Krull dimension of a ring K and of an extension R of K by considering modules of the form $N \otimes_K R$, where N is a simple or critical K-module.

1. INTRODUCTION

Let R be an Ore or a skew-Laurent extension of a ring K of finite Krull dimension in the sense of [3] or [5]. We investigate the Krull dimension of R-modules M of the form $N \otimes_K R$, where N is a finitely generated simple (or critical) K-module. Our basic result is that if N is simple, then $N \otimes R$ is either simple or 1-critical and we give precise conditions for either of these possibilities to occur.

Let $\mathscr{K}(M)$ denote the Krull dimension of M. We relate the conditions $\mathscr{K}(N \otimes_K R) = 0$ (respectively $= 1$) for all simple K-modules N to other conditions which have already appeared in the literature, viz., (i) no simple R-module has finite length over K and (ii) every simple R-module has finite length over K. (See Theorems 5.1 and 6.1 for the exact statements.)

2. PRELIMINARIES

Convention. From Section 4 onwards K is a right noetherian ring whose (right) Krull dimension is finite. Unless otherwise specified, R denotes an Ore or skew-Laurent extension of K and T the subring of R defined below. $\otimes$ denotes $\otimes_K$.

Recall that if d is a derivation of K, then each element of the Ore extension $R = K[x; d]$ can be written uniquely as $\sum_{i=0}^{n} a_i x^i$ (for some n) and in R, $xa - ax = d(a)$ for $a \in K$. If θ is an automorphism of K, let $T = K[x; \theta]$ denote the ring of twisted polynomials in which $ax = x\theta(a)$ for

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all $a \in K$. T has a right and left quotient ring with respect to the set of non-zero-divisors $\{x^n \mid n \geqslant 1\}$. This quotient ring is written $K[x, x^{-1}; \theta]$ and is called a skew-Laurent extension of K.

In either the d or θ cases, R is a free left (or right) K-module with the powers of x as a free basis.

We use the notion of Krull dimension as in [3] or [5]. If K is right noetherian, then $K[x; d]$, $K[x; \theta]$ and hence $K[x, x^{-1}; \theta]$ are all right noetherian and if K has finite Krull dimension, then the Krull dimension of each of these rings is either $\mathscr{K}(K)$ or $\mathscr{K}(K) + 1$. (This result is generalised in Theorem 4.7.)

3. Simple K-modules

Let I be a right ideal of K. Then $K/I \otimes R \cong R/IR$ (as R-modules). If $\sum a_i x^i \in R$, its image in R/IR is written $(\sum a_i x^i)^-$. Thus $(\sum a_i x^i)^- = (\sum b_i x^i)^-$ if and only if $a_i - b_i \in I$ for all i.

d-case

Let R be an Ore extension of K and I a maximal right ideal of K.

Lemma 3.1. *If for all $a \in K$ and all $n \in \mathbb{N}$, $(nd + a)I \not\subset I$, then IR is a maximal right ideal of R.*

Proof. This is a slight restatement of Lemma 17 of [2] with exactly the same proof. ∎

Conversely we have

Lemma 3.2. *If there exists an $a \in K$ and an $n \in \mathbb{N}$ with $(nd + a)I \subset I$ and n is not a zero-divisor on K/I, then R/IR has infinite length as an R-module.*

Proof. We define a map $\phi: R/IR \to R/IR$ by $\phi(\sum b_i x^i)^- = (\sum (nx + a)\, b_i x^i)^-$. If $b \in I$, then $(nx + a)b \in IR$ and so ϕ is well defined. ϕ is clearly a homomorphism of right R-modules. The condition on n implies that $\deg \phi(f) > \deg(f)$ (where deg denotes degree in x). Thus ϕ is injective but not surjective and hence R/IR must have infinite length. ∎

Remark 3.3. If K is an algebra over the rationals, then the condition in Lemma 3.2 can be rewritten as $(d + a)I \subset I$ for some $a \in K$. A more natural way of looking at this condition is the following. We say that two derivations of K are similar if their difference is an inner derivation of K. If d and d' are similar, $d' = d + \mathrm{ad}(a)$ say, then $K[x; d]$ coincides with $K[x'; d']$, where $x' = x + a$. Now the condition on I above can be rewritten as

$(d + \operatorname{ad} a)I \subset I$, i.e., "$I$ is stable under a derivation similar to d." When K is commutative this reduces to "I is stable under d."

θ-case

In this case, twisted R-module homomorphisms arise and we first develop this theory. Let S be a ring and θ an automorphism of S. If M and N are right S-modules, an abelian group homomorphism $\phi: M \to N$ is called a *θ-twisted S-module map* if $\phi(ms) = \phi(m)\,\theta(s)$. Clearly $\operatorname{Ker}\phi$ and $\operatorname{Image}\phi$ are S-modules. If M is a free S-module with free generators m_i and we choose $n_i \in N$, then there exists a unique θ-twisted S-module homomorphism ϕ satisfying $\phi(m_i) = n_i$.

If θ is an automorphism of K and $T = K[x; \theta]$, $R = K[x, x^{-1}, \theta]$, then θ can be extended to an automorphism of R (or T) by $\theta(\sum a_i x^i) = \sum \theta(a_i) x^i$. Then $x\theta(r) = rx$ for all $r \in R$. Now let S be any ring, θ an automorphism of S and suppose there exists a regular element $x \in S$ such that $sx = x\theta(s)$. If M is a right S-module, then right multiplication by x is a θ-twisted S-module homomorphism from M to M.

Let R be a skew-Laurent extension of K and I be a maximal right ideal of K.

LEMMA 3.4. *If for all $a \in K \backslash I$, $a\theta^n(I) \not\subset I$, then IR is a maximal right ideal of R.*

Proof (a minimal length argument). Suppose to the contrary that $IR \subset P \subset R$. Each element of R can be written as $(a_0 + a_1 x + \cdots + a_n x^n) x^m$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$, where a_0 and a_n are non-zero. n will be called the length of this element.

Among the elements of $P \backslash IR$ we choose one, y say, of least length and of the form $a_0 + \cdots + a_n x^n$. Then $a_0 \notin I$ and $a_n \notin I$. Since $a_n K + I = K$, we may also suppose that $a_n = 1$ and since $P \neq R$ we must have $n \geqslant 1$. For any $b \in I$, length $(by - y\theta^n(b)) < n$, hence $by - y\theta^n(b) \in IR$ and so $y\theta^n(b) \in IR$. If a_0 is the constant term of y, then $a_0 \theta^n(b) \in I$, i.e., $a_0 \theta^n(I) \subset I$. ∎

Conversely we have

LEMMA 3.5. *If there exists an $a \in K \backslash I$ and $n \in \mathbb{Z}$ such that $a\theta^n(I) \subset I$, then R/IR has infinite length as an R-module.*

Proof. We define a map $\phi: R/IR \to R/IR$ by $\phi(\sum_i d_i x^i)^- = (\sum_i (x^n + a)\, \theta^n(d_i) x^i)^-$. To check that ϕ is well defined we note that since $x\theta(d) = dx$, $(x^n + a)\, \theta^n(d) x^i = a\theta^n(d) x^i + dx^{n+i}$, and thus if $d \in I$, then $(x^n + a)\, \theta^n(d) x^i \in IR$.

Claim. ϕ is a θ^n-twisted R-module homomorphism. Since ϕ commutes with right multiplication by x and θ^n fixes x, it is sufficient to show that for

$b \in K$ $\phi((dx^i)^- b) = (\phi(dx^i)^-)\,\theta^n(b)$, and this is a straightforward computation using only that θ^i and θ^n commute. Since $\deg \phi(f) > \deg(f)$, ϕ is injective but not surjective and so R/IR is not of finite length over R. ∎

Remark 3.6. The condition in Lemma 3.5 can be rewritten as $K/\theta^n(I) \cong K/I$ (as K-modules via the map $\bar{b} \mapsto \overline{ab}$). So if I is a two-sided ideal (e.g., if K is commutative), then $\theta^n(I) = I$.

Two automorphism ψ and ϕ of K will be called similar if there exist units u, v in K such that $\psi = i(v)\,\phi i(u)$, where $i(u), i(v)$ are inner automorphisms of K, $i(u)(a) = u^{-1}au$. If ψ and ϕ are similar automorphisms, then $K[x, x^{-1}; \phi]$ coincides with $K[y, y^{-1}; \psi]$, where $y = uxv$ and if I is a maximal right ideal of K with $\phi(I) = I$, then $K/I \cong K/\psi(I)$. The analogous remark to Remark 3.3 for the automorphism case would be that if I is a maximal right ideal of K and ψ is an automorphism of K such that $K/I \cong K/\psi(I)$, then I is stable under some automorphism similar to ψ. Although this seems most unlikely we have not found a counterexample.

4. Associated Graded Arguments and Critical Modules (Recall Our Convention Stated in Section 2)

We assume that the reader is familiar with the standard arguments using filtered and associated graded methods. (See, e.g., [1; 5; 7, Sect. 3].) Let $R = K[x; d]$ or $K[x; \theta]$. R is a filtered ring by $R_n = \sum_{i=0}^{n} Kx^i$. The associated graded ring $\operatorname{Gr} R$ is isomorphic as a graded ring to $K[x]$ in the d-case and to $K[x; \theta]$ in the θ-case. If M is an R-module and M_0 a K-submodule which generates M, then M is a filtered R-module by $M_n = M_0 R_n$ and then $\operatorname{Gr} M$ denotes the associated graded module over $\operatorname{Gr} R$. In the particular case $M = N \otimes R$ we choose M_0 to be N.

Lemma 4.1. *If N is a K-module, then there is an isomorphism of graded* $\operatorname{Gr} R$*-modules,*

$$N \otimes_K \operatorname{Gr} R \cong \operatorname{Gr}(N \otimes R).$$

Proof. Since $R_n = R_{n-1} \oplus Kx^n$, $N \otimes R_n / N \otimes R_{n-1} \cong N \otimes R_n / R_{n-1}$. ∎

Definition 4.2. The lattice of graded $\operatorname{Gr} R$-submodules of $N \otimes \operatorname{Gr} R$ will be denoted by $\mathscr{L}_{\mathrm{gr}}(N \otimes \operatorname{Gr} R)$ and the Krull dimension of this lattice will be denoted by $\mathscr{K}_{\mathrm{gr}}(N \otimes \operatorname{Gr} R)$.

Lemma 4.3. *The map $\chi: \mathscr{L}(N \otimes R) \to \mathscr{L}_{\mathrm{gr}}(N \otimes \operatorname{Gr} R)$ given by $P \mapsto \operatorname{Gr} P$ is injective when it is restricted to a chain of submodules.*

Proof. Immediate, since the filtration on R is discrete. ∎

LEMMA 4.4. *Let N be a ρ-critical noetherian K-module and $R = K[x; \theta]$. Then $\mathscr{K}(N \otimes \operatorname{Gr} R) = \mathscr{K}_{\mathrm{gr}}(N \otimes \operatorname{Gr} R) = \mathscr{K}(N) + 1$ and $N \otimes \operatorname{Gr} R$ is $\rho + 1$-critical.*

Proof. Similar to the proof of [3, Proposition 9.3] which considers the case $\theta = Id$. ∎

LEMMA 4.5. *Let A, B be partially ordered sets and ϕ an order preserving map from A to B. Suppose that $[a_1, a_2] \neq 0$ implies that $\operatorname{dev}[\phi(a_1), \phi(a_2)] \geqslant m$. If B is τ-critical, then A is at most $(\tau - m)$-critical. (Compare 4.7(b) below.)*

Proof. Similar to [5, a]. ∎

LEMMA 4.6. *Let R be a skew-Laurent extension of K, T the subring $K[x; \theta]$ of R, N a K-module, and ψ the map $\mathscr{L}(N \otimes R) \to \mathscr{L}(N \otimes T)$ given by $P \mapsto P \cap (N \otimes T)$. Then ψ is injective.* ∎

THEOREM 4.7. *Let N be a noetherian K-module which is ρ-critical and R be an Ore or skew-Laurent extension of K.*

(a) $\mathscr{K}(N \otimes R) \geqslant \rho$.

(b) *$N \otimes R$ is at most $(\rho + 1)$-critical, i.e., if M is a non-zero R-submodule of $N \otimes R$, then $\mathscr{K}(N \otimes R/M) \leqslant \rho$.*

(c) *If no simple R-module has finite length over K, then $N \otimes R$ is ρ-critical.*

Proof. In the Ore case let

$$\phi = \chi\colon \mathscr{L}(N \otimes R) \to \mathscr{L} \equiv \mathscr{L}_{\mathrm{gr}}(N \otimes \operatorname{Gr} R), P \mapsto \operatorname{Gr} P$$

and in the skew-Laurent case let

$$\phi = \chi \circ \psi\colon \mathscr{L}(N \otimes R) \to \mathscr{L} \equiv \mathscr{L}_{\mathrm{gr}}(N \otimes \operatorname{Gr} T),$$
$$P \mapsto \operatorname{Gr}(P \cap (N \otimes T)).$$

(a) That $\mathscr{K}(N \otimes R) \geqslant \rho$ is clear, since ${}_K R$ is free.

(b) In both cases ϕ is injective on chains of submodules and Lemma 4.4 then gives the upper bound.

(c) By 4.4 and 4.5 it is sufficient to show that if P_1 and P_2 are submodules of $N \otimes R$, then

$$(*) \qquad \mathscr{K}_{\mathrm{gr}}(\phi(P_1)/\phi(P_2)) \geqslant 1.$$

Consider the skew-Laurent case and let G_1, G_2 be elements of $\mathscr{L}$ with $G_1 \subset G_2$. As in the Rentschler–Gabriel argument [5] we have

(1) If G_1/G_2 is simple over Gr T, then G_1/G_2 is simple over K.

(2) If G_1/G_2 is artinian over Gr T, then G_1/G_2 has finite length over Gr T and hence over K.

(3) If $M_1, M_2 \in \mathscr{L}(N \otimes T)$ with $M_1 \supset M_2$ and $\chi(M_1)/\chi(M_2)$ has finite length over K, then M_1/M_2 has finite length over K.

We need also

(4) If $P_1, P_2 \in \mathscr{L}(N \otimes R)$ with $P_1 \supset P_2$, then $\psi(P_1)/\psi(P_2)$ cannot have finite length over K.

To show this set $M_i = \psi(P_i)$ and suppose to the contrary that M_1/M_2 has finite length over K. Let L/M_2 be a simple T-submodule of M_1/M_2. x acts injectively on L/M_2 since if $yx \in M_2$, then $y \in P_2$ and so $y \in M_2$. Also $xT = Tx$ so $(L/M_2)x = L/M_2$. Thus x acts bijectively on L/M_2 and, by universality of localisation, L/M_2 is an R-module which has finite length over K, which contradicts the hypothesis. (*) follows immediately from (1) to (4).

In the Ore case, one uses versions of (1)–(3). ∎

COROLLARY 4.8. *Let R be either an Ore extension of K (where K is an algebra over the rationals) or a skew-Laurent extension of K. If N is a simple K-module then $N \otimes R$ is either simple or* 1*-critical.*

Proof. See Lemmas 3.1, 3.2, and 3.5 and Remark 3.3.

5. SIMPLE MODULES ENLARGE

THEOREM 5.1. *Consider the following conditions,*

(a) *All simple R-modules have finite length over K.*

(b) *If N is any simple K-module, then $N \otimes R$ is* 1*-critical over R.*

(c) *For all $\rho \geqslant 0$, if N is a noetherian ρ-critical K-module, then $N \otimes R$ is $(\rho + 1)$-critical over R.*

(d) $\mathscr{K}(R) = \mathscr{K}(K) + 1$.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d), (d) $\not\Rightarrow$ (b), (b) $\not\Rightarrow$ (a).

Proof. (a) $\Rightarrow$ (b) Note first that no non-zero R-submodule of $N \otimes R$ can be finitely generated over K. Hence given (a), $N \otimes R$ has no simple R-submodules and (b) follows by Section 3 and Corollary 4.8.

(b) $\Rightarrow$ (c) follows by a straightforward induction on ρ (cf. [6, Lemma 10]).

(c) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious.

(b) $\not\Rightarrow$ (a) even for commutative extensions, i.e., when $d=0$ or $\theta = Id$. Let K be the local ring $k[y]_{(y)}$ and R be $K[x]$ or $K[x, x^{-1}]$. Then (b) holds. In either case consider $R/(1-yx)R \cong k(y)$ (as K-modules) This is a simple R-module which is torsion free as a K-module.

(d) $\not\Rightarrow$ (b) Take $K = k[y]$, k algebraically closed, with $d(y) = y$ or $\theta(y) = \lambda y$, λ not a root of unity. In either case (y) is the only maximal ideal of K stable under d or any power of θ. So there is only one simple K-module N such that $N \otimes R$ is 1-critical over R but $\mathscr{K}(R/(y)) = 1$ and $\mathscr{K}(R) = 2$. ∎

(For the skew-Laurent case most of the above theorem is due to Segal [6, Lemma 10]. In condition (a) he replaced finite length by artinian but the stronger condition is needed here, since in the Ore case it is possible for a simple R-module of the form $N \otimes R$ to be artinian over K.)

6. Simple Modules Stay Simple

THEOREM 6.1. *Consider the following conditions.*

(α) *No simple R-module is of finite length over K.*

(β) *If N is any simple K-module, then $N \otimes R$ is a simple R-module.*

(γ) *For all $\rho \geqslant 0$, if N is a ρ-critical noetherian K-module, then $N \otimes R$ is a ρ-critical R-module.*

(δ) $\mathscr{K}(R) = \mathscr{K}(K)$.

Then (α) $\Leftrightarrow$ (β) $\Leftrightarrow$ (γ) $\Rightarrow$ (δ). (δ) $\not\Rightarrow$ (α).

Proof. (α) $\Rightarrow$ (β) Suppose not and let I be a maximal right ideal of K and P a right ideal of R with $R \supset P \supset IR$. By the usual length argument, P contains a monic element $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ and in the skew-Laurent case, P also contains a "comonic element" of the form $1 + b_1 x + \cdots + b_n x^n$. Thus R/P is the image of $\sum_{i=0}^{n} K/I \otimes x^i$ and so R/P is of finite length over K, a contradiction.

(β) $\Rightarrow$ (α) Suppose that P is a maximal right ideal of R with $(R/P)_K$ of finite length. Clearly $K/K \cap P$ has finite length and since R/P is simple the canonical homomorphism from $K/K \cap P \otimes R$ into R/P is surjective. By (β) the simple subquotients of $K/K \cap P \otimes R$ are of the form $N \otimes R$ where N_K is simple. Thus $R/P \cong N \otimes R$ for some such N. Since $N \otimes R$ cannot have finite length over K we have a contradiction.

$(\alpha) \Rightarrow (\gamma)$ by Theorem 4.7.

$(\gamma) \Rightarrow (\beta)$ and $(\gamma) \Rightarrow (\delta)$ are obvious.

$(\delta) \not\Rightarrow (\alpha)$ Let k be an algebraically closed field of characteristic zero. We first construct a certain localisation of the commutative polynomial ring $k[u, v]$. For $n \in \mathbb{N}$, let M_n be the maximal ideal $(u-1, v-n)$ and for $m \in 2\mathbb{N}$ let $P_m = (u+m)$.

Let S be the set of elements not belonging to an M_n or a P_m.

If M is a maximal ideal containing a P_m, then $M \cap S \neq 0$ (Suppose $M = (u+m, v+\lambda)$ say. If $\lambda \notin \mathbb{Z}$, then $v+\lambda \in S$. So suppose $\lambda = p \in \mathbb{Z}$. Set $z = \frac{1}{2}(u+m) + (v+p) \in M$; the residue of z in K/M_n is $\frac{1}{2}(1+m) + (n+p) \neq 0$.)

Set $K = k[u, v]_S$. Then the ideals P_m generate maximal ideals of K which we again denote by P_m.

If d is the derivation of K defined by $d(u) = 0$, $d(v) = 1$, then $K/P_m \otimes R$ is 1-critical for all m. R is a localisation of $k[x, v][u]$ and hence $\mathscr{K}(R) = 2 = \mathscr{K}(K)$. If θ is the automorphism of K defined by $\theta(u) = u$, $\theta(v) = v+1$ and $R = K[x, x^{-1}; \theta]$, then $K/P_m \otimes R$ is 1-critical for all m. R is a localisation of $k[v, x, x^{-1}][u]$ and hence $\mathscr{K}(R) = 2 = \mathscr{K}(K)$. ∎

COROLLARY 6.2. *Let K be commutative. Suppose either that K is an algebra over the rationals and no maximal ideal of K is stable under d or that no maximal ideal of K is stable under a power of θ. Then $\mathscr{K}(R) = \mathscr{K}(K)$.*

Proof. By Section 3, if N_K is simple then $N \otimes R$ is simple. ∎

COROLLARY 6.3. *Suppose that K is an algebra over the rationals and that no maximal right ideal of K is stable under a derivation similar to d. Then $\mathscr{K}(R) = \mathscr{K}(K)$.*

7. CONCLUDING REMARKS

If K is a commutative noetherian ring and $R = K[x]$, then $\{N \otimes R \mid N_K$ finitely generated$\}$ generates the family of all finitely generated R-modules in the sense of [4, Theorem 183]. It would be interesting to know whether there is a corresponding result for the more general K and R considered here.

Let $\mathbb{N}$ have the ordering $1 > 2 > 3 > \cdots$ and let $\mathbb{N}^n$ have the corresponding lexicographic ordering. If K has Krull dimension n, then K has a descending chain of right ideals $\mathscr{C} = \{I_i \mid i \in \mathbb{N}^n\}$, where $I_i \supset I_j$ if $i > j$. (So for $n = 2$ we have $I_{11} \supset I_{12} \supset \cdots \supset I_{21} \supset I_{22} \supset \cdots \supset I_{31} \supset \cdots$.)

Conjecture. Let K have Krull dimension n and R be an Ore or skew-Laurent extension of K. Then $\mathscr{K}(R) = \mathscr{K}(K) + 1$ if and only if there is some

such chain $\mathscr{C}$ such that if I, J are successive terms of the chain, then $\mathscr{K}(I/J \otimes R) \geqslant 1$. (The "if" part is obvious.)

Note added in proof. There is some overlap between the results of this paper and those of a recent preprint "Krull dimension of differential operator rings" by K. R. Goodearl and R. B. Warfield Jr. but the methods used in the two papers are rather different. Their paper contains a necessary and sufficient condition for $\mathscr{K}(R) = \mathscr{K}(K)$, where $R = K[x; d]$ and K is a commutative noetherian ring of finite Krull dimension.

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