

DEGENERATIONS AND REPRESENTATIONS OF TWISTED SHIBUKAWA-UENO R -OPERATORS

ROBIN ENDELMAN AND TIMOTHY J. HODGES

ABSTRACT. We study degenerations of the Belavin R -matrices via the infinite dimensional operators defined by Shibukawa-Ueno. We define a two-parameter family of generalizations of the Shibukawa-Ueno R -operators. These operators have finite dimensional representations which include Belavin's R -matrices in the elliptic case, a two-parameter family of twisted affinized Cremmer-Gervais R -matrices in the trigonometric case, and a two-parameter family of twisted (affinized) generalized Jordanian R -matrices in the rational case. We find finite dimensional representations which are compatible with the elliptic to trigonometric and rational degeneration. We further show that certain members of the elliptic operators have no finite dimensional representations. These R -operators unify and generalize earlier constructions of Felder and Pasquier, Ding and Hodges, and the authors, and illuminate the extent to which the Cremmer-Gervais R -matrices (and their rational forms) are degenerations of Belavin's R -matrix.

1. INTRODUCTION

In this article, we give a unified description of finite dimensional representations of twisted Shibukawa-Ueno R -operators. In [11], Shibukawa and Ueno described a set of solutions of the Yang-Baxter equation on the field of meromorphic functions on \mathbb{F} . These solutions were of three types, elliptic, trigonometric and rational. In [6], Felder and Pasquier showed that the elliptic solutions could be twisted and restricted to a finite dimensional subspace in such a way that they produced Belavin's solutions of the Yang-Baxter equation. Ding and the second author observed in [4] that a similar procedure applied to the analogous trigonometric solutions of the constant Yang-Baxter equation yielded the Cremmer-Gervais R -matrices. The authors then showed in [5] that this procedure applied to the constant rational solutions yielded a generalization of the Jordanian R -matrix which quantized certain solutions of the classical Yang-Baxter equation studied by Gerstenhaber and Giaquinto in [7, 8]. Our aim here is to unify and generalize these constructions in order to explain to what extent the Belavin R -matrices degenerate into affinized Cremmer-Gervais R -matrices and their rational analogs.

First we observe that a simple twisting procedure enables us to extend the Shibukawa-Ueno operators to a two parameter family of Yang-Baxter operators. We then look for finite dimensional representations of these operators which are compatible with their degeneration. In the elliptic case, only certain members of this family have such finite dimensional representations and these representations are essentially those studied by Felder and Pasquier. In the trigonometric case the

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whole family has a simultaneous finite dimensional representation and the family of R -matrices obtained is a 2-parameter family of deformations of the affinized Cremmer-Gervais R -matrices constructed using an analogous twisting mechanism. The Belavin R -matrices then degenerate naturally not to the Cremmer-Gervais matrices themselves but to other members of this family. This is analogous to the fashion in which Antonov, Hasegawa and Zabrodin realised the Cremmer-Gervais matrices as twists of degenerations of the Belavin matrices [1]. In the rational case, the generalized Shibukawa-Ueno operators also have a simultaneous finite dimensional representation which yields a two parameter family of deformations of the (affinized) generalized Jordanian R -matrices constructed by the authors in [5].

In order to obtain the required degenerations of the finite dimensional matrices, we degenerate the finite dimensional subspaces in a fashion consistent with the degeneration of the operators. Thus the subspace on which we represent the elliptic operator has a basis of elliptic functions, while the subspace on which we represent the trigonometric operators and rational operators have bases of trigonometric and rational functions, respectively.

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2. BACKGROUND

2.1. Yang-Baxter equation on fields of meromorphic functions. We begin by setting out an appropriate context in which to discuss the Yang-Baxter equation on function fields. Denote the field of meromorphic functions on \mathbb{C}^n by $\mathcal{M}(\mathbb{C}^n)$. In general it is not possible to extend a linear operator $\mathcal{R} \in \text{End}(\mathcal{M}(\mathbb{C}^2))$ to an operator $\mathcal{R}_{12} \in \text{End}(\mathcal{M}(\mathbb{C}^3))$. Thus we must work inside a subring of $\text{End}(\mathcal{M}(\mathbb{C}^2))$ in which this is possible.

Denote by $\text{Aut}(\mathbb{C}^n)$ the group of automorphisms of \mathbb{C}^n . For any function $\phi \in \text{Aut}(\mathbb{C}^n)$ define $\phi^* \in \text{End}(\mathcal{M}(\mathbb{C}^n))$ by $\phi^*(f) = f \circ \phi$. Set $G(n) = \{\phi^* | \phi \in \text{Aut}(\mathbb{C}^n)\}$, and let $\mathcal{A}(n)$ denote the subalgebra of $\text{End}(\mathcal{M}(\mathbb{C}^n))$ generated by $G(n)$ over the subfield $\mathcal{M}(\mathbb{C}^n)$ (acting as multiplication operators). It is easily verified that $G(n)$ is linearly independent over $\mathcal{M}(\mathbb{C}^n)$, and that $\mathcal{M}(\mathbb{C}^n)$ is a $G(n)$ -module algebra. For $\phi^* \in G(n)$ and $f \in \mathcal{M}(\mathbb{C}^n)$, $\phi^*.f = \phi^*(f)\phi^*$, and so $\mathcal{A}(n)$ is isomorphic to the smash product $\mathcal{M}(\mathbb{C}^n) \# G(n)$. Denote $\mathcal{A}(2)$ by \mathcal{A} .

For $\phi \in \text{Aut}(\mathbb{C}^2)$, define $\phi_{ij} \in \text{Aut}(\mathbb{C}^3)$ by ϕ acting on the i^{th} and j^{th} variables. Then $(\phi^*)_{ij} = (\phi_{ij})^* \in G(3)$, and for $\mathcal{R} = \sum_{\alpha} f_{\alpha}(z_1, z_2)\phi_{\alpha}^* \in \mathcal{A}$, with $f_{\alpha} \in \mathcal{M}(\mathbb{C}^2)$ and $\phi_{\alpha}^* \in G(2)$, we may define $\mathcal{R}_{ij} \in \mathcal{A}(3)$ by

$$\mathcal{R}_{ij} = \sum_{\alpha} f_{\alpha}(z_i, z_j)(\phi_{\alpha}^*)_{ij}.$$

Definition 2.1. A solution, \mathcal{R} , in \mathcal{A} to the Yang-Baxter equation will be called an R -operator. As in the case for R -matrices, spectral parameter-dependent R -operators are maps $\mathcal{R} : \Omega \rightarrow \mathcal{A}$, where $\Omega \subset \mathbb{C}^k$ with $k = 1$ or 2 , satisfying the spectral parameter-dependent Yang-Baxter equation

$$\mathcal{R}_{12}(\lambda_1, \lambda_2)\mathcal{R}_{13}(\lambda_1, \lambda_3)\mathcal{R}_{23}(\lambda_2, \lambda_3) = \mathcal{R}_{23}(\lambda_2, \lambda_3)\mathcal{R}_{13}(\lambda_1, \lambda_3)\mathcal{R}_{12}(\lambda_1, \lambda_2).$$

Denote by P the R -operator, in $G(2)$, which acts on $\mathcal{M}(\mathbb{C}^2)$ by $P \cdot f(z_1, z_2) = f(z_2, z_1)$.

2.2. The Shibukawa-Ueno R -operators. In [11], Shibukawa and Ueno constructed a family of operators $\mathcal{R}^\theta(\lambda) \in \mathcal{A}$, depending on a holomorphic function θ :

$$\mathcal{R}^\theta(\lambda) = G_\theta(z_1 - z_2, \lambda)P - G_\theta(z_1 - z_2, \kappa)I,$$

where $G_\theta(z, \lambda) = \frac{\theta'(0)\theta(z + \lambda)}{\theta(z)\theta(\lambda)}$ and I is the identity operator. These operators satisfy the Yang-Baxter equation for any scalar κ and any function θ satisfying the ‘‘three-term equation’’:

$$\sum_{\substack{\text{cyc perms} \\ \text{of } y, z, w}} \theta(x + y)\theta(x - y)\theta(z + w)\theta(z - w) = 0.$$

Analytic solutions of the three-term equation are [12] $\theta(z) = Ae^{Bz^2}\vartheta_1(Cz, \tau)$ and its trigonometric and rational degenerations, where A, B, C are constants, $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}z > 0\}$, and ϑ_1 is Jacobi’s first theta function:

$$\vartheta_1(z, \tau) = - \sum_{m \in \mathbb{Z}} \exp \left\{ \pi i \left(m + \frac{1}{2} \right)^2 \tau + 2\pi i \left(m + \frac{1}{2} \right) \left(z + \frac{1}{2} \right) \right\}.$$

Here we will consider the particular cases when $A = 1$ and $B = 0$. Specifically, we consider the following families of R -operators:

Elliptic case: $\mathcal{R}^e(\lambda)$ with $\theta(z) = \vartheta_1(z, \tau)$, $\tau \in \mathbb{H}$

Trigonometric case: $\mathcal{R}^t(\lambda)$ with $\theta(z) = \sin \pi z / \tau_1$, $\tau_1 \in \mathbb{C} \setminus \{0\}$

Rational case: $\mathcal{R}^r(\lambda)$ with $\theta(z) = z$.

In [11], Shibukawa and Ueno also introduce the notion of obtaining R -matrices by restriction of $\mathcal{R}^\theta(\lambda)$ to finite dimensional invariant subspaces. For $n < \infty$, let $V = \bigoplus_a \mathbb{C}f_a$ be an n -dimensional subspace of $\mathcal{M}(\mathbb{C})$, with basis $\{f_a \mid a = 0, 1, \dots, n-1\}$, and identify $V \otimes V$ with the space of functions in 2 variables $\text{Span}_{\mathbb{C}}\{f_a(z_1)f_b(z_2)\}$.

Definition 2.2. [11] If $\mathcal{R} \in \mathcal{A}$ is a solution of the Yang-Baxter equation and $V \otimes V$ is invariant under \mathcal{R} , then we say that $R|_{V \otimes V} \in \text{End}(V \otimes V)$ is a *finite dimensional representation of \mathcal{R}* . For a spectral parameter-dependent R -operator, $\mathcal{R} : \Omega \rightarrow \mathcal{A}$, we say that $R|_{V \otimes V}$ is a finite dimensional representation of \mathcal{R} if $V \otimes V$ is invariant under $\mathcal{R}(\lambda)$ for all $\lambda \in \Omega$.

If \mathcal{R} is an R -operator, then $\mathcal{R}|_{V \otimes V}$ is a matrix solution of the Yang-Baxter equation. In Section 4, we give finite dimensional representations for twisted Shibukawa-Ueno R -operators defined in Section 3, in each of the elliptic, trigonometric, and rational cases, and the corresponding degenerations.

3. TWISTED SHIBUKAWA-UENO OPERATORS

The twists that we shall be considering are all of one simple kind.

Theorem 3.1. *Let $R(\lambda) \in \text{End}(V \otimes V)$ be a solution of the Yang-Baxter equation. Let $B : (\mathbb{C}, +) \rightarrow GL(V)$ be a homomorphism such that $R(\lambda)$ commutes with $B(\mu) \otimes B(\mu)$ for all λ and μ . For $\alpha, \beta \in \mathbb{C}$, set*

$$F_{\alpha, \beta}(\lambda) = B(\alpha\lambda - \beta) \otimes B^{-1}(\alpha\lambda - \beta)$$

Then

$$R_{\alpha, \beta}(\lambda) = F_{\alpha, \beta}(-\lambda)R(\lambda)F_{\alpha, \beta}(\lambda)$$

also satisfies the Yang-Baxter equation.

Proof. It suffices to prove the result in two separate case: 1) when $\alpha = 1$ and $\beta = 0$ and 2) when $\alpha = 0$. Let B_i denote B acting in the i^{th} component of the tensor product.

Case 1: Suppose $\alpha = 1$ and $\beta = 0$. Since $R_{1,0}(\lambda) = B_1(-2\lambda)R(\lambda)B_1(2\lambda)$, it actually suffices to prove that $\tilde{R}(\lambda) = B_1(\lambda)R(\lambda)B_1(\lambda)$ satisfies the Yang-Baxter equation.

Now

$$\begin{aligned}
& \tilde{R}_{12}(\lambda - \lambda')\tilde{R}_{13}(\lambda)\tilde{R}_{23}(\lambda') \\
&= B_1(\lambda - \lambda')R_{12}(\lambda - \lambda')B_1(\lambda' - \lambda)B_1(\lambda)R_{13}(\lambda)B_1(-\lambda)B_2(\lambda')R_{23}(\lambda')B_2(-\lambda') \\
&= B_1(\lambda - \lambda')R_{12}(\lambda - \lambda')B_1(\lambda')B_2(\lambda')R_{13}(\lambda)R_{23}(\lambda')B_2(-\lambda')B_1(-\lambda) \\
&= B_1(\lambda)B_2(\lambda')R_{12}(\lambda - \lambda')R_{13}(\lambda)R_{23}(\lambda')B_2(-\lambda')B_1(-\lambda) \\
&= B_1(\lambda)B_2(\lambda')R_{23}(\lambda')R_{13}(\lambda)R_{12}(\lambda - \lambda')B_2(-\lambda')B_1(-\lambda) \\
&= B_1(\lambda)B_2(\lambda')R_{23}(\lambda')R_{13}(\lambda)B_1(-\lambda')B_2(-\lambda')R_{12}(\lambda - \lambda')B_1(\lambda' - \lambda) \\
&= B_2(\lambda')R_{23}(\lambda')B_2(-\lambda')B_1(\lambda)R_{13}(\lambda)B_1(-\lambda)B_1(\lambda - \lambda')R_{12}(\lambda - \lambda')B_1(\lambda' - \lambda) \\
&= \tilde{R}_{23}(\lambda')\tilde{R}_{13}(\lambda)\tilde{R}_{12}(\lambda - \lambda')
\end{aligned}$$

Case 2: Suppose $\alpha = 0$. Since $R_{0,\beta}(\lambda) = B_1(-2\beta)R(\lambda)B_2(2\beta)$, it suffices to prove that $\tilde{R}(\lambda) = B_1(\beta)R(\lambda)(B_2(-\beta))$ satisfies the Yang-Baxter equation. Now

$$\begin{aligned}
& \tilde{R}_{12}(\lambda - \lambda')\tilde{R}_{13}(\lambda)\tilde{R}_{23}(\lambda') \\
&= B_1(\beta)R_{12}(\lambda - \lambda')B_2(-\beta)B_1(\beta)R_{13}(\lambda)B_3(-\beta)B_2(\beta)R_{23}(\lambda')B_3(-\beta) \\
&= B_1(\beta)R_{12}(\lambda - \lambda')B_2(\beta)B_1(\beta)R_{13}(\lambda)B_3(-\beta)B_2(-\beta)R_{23}(\lambda')B_3(-\beta) \\
&= B_1(2\beta)B_2(\beta)R_{12}(\lambda - \lambda')R_{13}(\lambda)R_{23}(\lambda')B_2(-\beta)B_3(-2\beta) \\
&= B_1(2\beta)B_2(\beta)R_{23}(\lambda')R_{13}(\lambda)R_{12}(\lambda - \lambda')B_2(-\beta)B_3(-2\beta) \\
&= B_2(\beta)R_{23}(\lambda')B_1(2\beta)R_{13}(\lambda)B_3(-2\beta)R_{12}(\lambda - \lambda')B_2(-\beta) \\
&= B_2(\beta)R_{23}(\lambda')B_3(-\beta)B_1(\beta)R_{13}(\lambda)B_3(-\beta)B_1(\beta)R_{12}(\lambda - \lambda')B_2(-\beta) \\
&= \tilde{R}_{23}(\lambda')\tilde{R}_{13}(\lambda)\tilde{R}_{12}(\lambda - \lambda')
\end{aligned}$$

□

Suppose that V has a basis e_i , for $i = 1, \dots, n$. An operator $R(\lambda) \in \text{End}(V \otimes V)$ is said to be homogeneous if it is of the form

$$R(\lambda)(e_i \otimes e_j) = \sum_k a_k(\lambda)e_k \otimes e_{i+j-k}$$

The diagonal operator $B(\mu)e_k = \exp(c\mu k)e_k$ satisfies the hypothesis of the theorem for any homogeneous operator $R(\lambda)$ and the resulting twisted operator is

$$R_{\alpha,\beta}(\lambda)(e_i \otimes e_j) = \sum_k \exp\{2c[\alpha\lambda(i-k) - \beta(k-j)]\}a_k(\lambda)e_k \otimes e_{i+j-k}.$$

The Cremmer-Gervais R -matrices [3] are homogeneous, given by

$$(R_{CG})_{ij}^{kl} = p^{2(j-k)} \begin{cases} q & i = k \geq j = l, \\ q^{-1} & i = k < j = l, \\ -\hat{q} & i < k < j, i + j = k + l, \\ \hat{q} & j \leq k < i, i + j = k + l, \\ 0 & \text{otherwise,} \end{cases}$$

where $p^n = q$ and $\hat{q} = q - q^{-1}$. The 2-parameter versions (p and q independent) described in [9] are constructed from twists of this form ($\alpha = 0$).

The corresponding solution of the braid equation \check{R}_{CG} satisfies the Hecke relation $(\check{R}_{CG} - q)(\check{R}_{CG} + q^{-1}) = 0$ and as such, R_{CG} has a standard affinization [10, page 296] given by $R_{CG}(\lambda) = \hat{q}\eta P - \hat{\eta}R_{CG}$, where $\eta = e^{\pi i \lambda}$, whose matrix coefficients are given by

$$R_{CG}(\lambda)_{ij}^{kl} = p^{2(j-k)} \begin{cases} \widehat{q\eta^{-1}} & i = j = k = l, \\ -\hat{\eta}q^{\text{sgn}(i-j)} & i = k \neq l = j, \\ \text{sgn}(j-i)\hat{\eta}\hat{q} & \min(i, j) < k < \max(i, j), i + j = k + l, \\ \hat{q}\eta^{\text{sgn}(j-i)} & j = k \neq l = i, \\ 0 & \text{otherwise.} \end{cases}$$

With $B(\mu)e_k = e^{2\pi i \mu k}e_k$, we define a two-parameter family of twisted Cremmer-Gervais operators by

$$(3.1) \quad R_{CG(\alpha, \beta)}(\lambda) = F_{\alpha, \beta}(-\lambda)R_{CG}(\lambda)F_{\alpha, \beta}(\lambda).$$

The matrix coefficients are then given by

$$R_{CG(\alpha, \beta)}(\lambda)_{ij}^{kl} = \zeta^{2(i-k)}\gamma^{2(j-k)}R_{CG}(\lambda)_{ij}^{kl}$$

where $\zeta = e^{2\pi i \alpha \lambda}$ and $\gamma = e^{2\pi i \beta}$.

An analogous version of the twisting theorem for operators on function spaces is the following.

Theorem 3.2. *Let $\mathcal{R}(\lambda) \in \mathcal{A}$ be an \mathcal{R} -operator. Let $\phi : \mathbb{C} \rightarrow \text{Aut}(\mathbb{C})$ be such that $\mathcal{R}(\lambda)$ commutes with $\phi^*(\mu) \otimes \phi^*(\mu)$ for all λ and μ . For $\alpha, \beta \in \mathbb{C}$, set*

$$\mathcal{F}_{\alpha, \beta}(\lambda) = \phi^*(\alpha\lambda - \beta) \otimes (\phi^*)^{-1}(\alpha\lambda - \beta)$$

Then

$$\mathcal{R}_{\alpha, \beta}(\lambda) = \mathcal{F}_{\alpha, \beta}(-\lambda)\mathcal{R}(\lambda)\mathcal{F}_{\alpha, \beta}(\lambda)$$

also satisfies the Yang-Baxter equation.

Proof. Analogous to the proof in the finite dimensional case. \square

Define $\phi : (\mathbb{C}, +) \rightarrow \text{Aut}(\mathbb{C})$ by $\phi(\lambda) \cdot z = z + \lambda$. Define also operators $\tilde{\mathcal{F}}_s \in \mathcal{A}$ by $\tilde{\mathcal{F}}_s \cdot f(z_1, z_2) = f(z_1 + s, z_2 - s)$. If $\mathcal{R}^\theta(\lambda)$ is the Shibukawa-Ueno R -operator defined above, then it is easily seen that $\mathcal{R}^\theta(\lambda)$ commutes with $\phi^*(\mu) \otimes \phi^*(\mu)$ and hence,

$$\begin{aligned} \mathcal{R}_{\alpha, \beta}^\theta(\lambda) &= \mathcal{F}_{\alpha, \beta}(-\lambda)\mathcal{R}^\theta(\lambda)\mathcal{F}_{\alpha, \beta}(\lambda) \\ &= G(z_1 - z_2 - 2(\alpha\lambda + \beta), \lambda)\tilde{\mathcal{F}}_{-2\alpha\lambda}P - G(z_1 - z_2 - 2(\alpha\lambda + \beta), \kappa)\tilde{\mathcal{F}}_{-2\beta} \end{aligned}$$

satisfies the Yang-Baxter equation.

Remark 1. Taking $\alpha = 1/2n$ and $\beta = \kappa/2n$ ($n \in \mathbb{Z}^\times$), the resulting twisted R -operator in the elliptic case is the same as that of Felder and Pasquier (in [6], they twist the elliptic Shibukawa-Ueno R -operator by a two-parameter $F(\lambda_1, \lambda_2)$).

Taking $\alpha = 0$ and $\beta = \kappa/2n$, $\mathcal{F}_{0, \kappa/2n}$ is the constant twist applied to $\mathcal{R}_\infty^t (= \lim_{\text{im } \lambda \rightarrow \infty} \mathcal{R}^t(\lambda))$ in [4] in the trigonometric case, and to $\mathcal{R}_\infty^r (= \lim_{\lambda \rightarrow \infty} \mathcal{R}^r(\lambda))$ in [5] in the rational case.

4. FINITE DIMENSIONAL REPRESENTATIONS AND DEGENERATIONS

4.1. Elliptic case. Let $\mathcal{H}(\mathbb{C}^n)$ denote the space of holomorphic functions on \mathbb{C}^n . Fix $\tau \in \mathbb{H}$, the upper half-plane, and let

$$(4.1) \quad V_n^e \stackrel{\text{def}}{=} \{f \in \mathcal{H}(\mathbb{C}) \mid f(z+1) = (-1)^{n-1}f(z), f(z+\tau) = e^{-\pi i n \tau - 2\pi i n z} f(z)\}.$$

Proposition 4.1. *The vector space V_n^e has basis $\mathcal{B} = \{\psi_a(z) \mid a \in \mathbb{Z}_n\}$ where*

$$\psi_a(z) = \sum_{\substack{m \in \mathbb{Z} \\ m \equiv a \pmod{n}}} e^{\pi i (m - \frac{n-1}{2})^2 \tau / n + 2\pi i (m - \frac{n-1}{2}) z}.$$

Proof. It is well known that $\dim(V_n^e) = n$, so it suffices to show that \mathcal{B} spans V_n^e . Let $f \in V_n^e$, and set $g(z) = e^{\pi i (n-1)z} f(z)$. Since $f(z+1) = (-1)^{n-1}f(z)$, g satisfies $g(z+1) = g(z)$, and has Laurent expansion $g(z) = \sum_{m \in \mathbb{Z}} a_m e^{2\pi i m z}$. Thus

$$f(z) = e^{-\pi i (n-1)z} g(z) = \sum_{m \in \mathbb{Z}} a_m e^{2\pi i (m - \frac{n-1}{2})z}.$$

Now, writing $a_m = c_m e^{\pi i (m - \frac{n-1}{2})^2 \tau / n}$, from the quasi-periodicity of f with respect to τ , we obtain $c_m = c_{m+n}$ for all $m \in \mathbb{Z}$. Hence, $f(z) = \sum_{a \in \mathbb{Z}_n} c_a \psi_a(z)$. \square

Identify $V_n^e \otimes V_n^e$ with the function space $\text{Span}_{\mathbb{C}}\{\psi_a(z_1)\psi_b(z_2)\} = \{f \in \mathcal{H}(\mathbb{C}^2) \mid f(-, z_2), f(z_1, -) \in V_n^e\}$. Let $\theta_{a,b}$ denote the standard theta functions of rational characteristic, defined, for $a, b \in \mathbb{Q}$, by:

$$\theta_{a,b}(z, \tau) = \sum_{m \in \mathbb{Z}} \exp\left\{\pi i (m+a)^2 \tau + 2\pi i (m+a)(z+b)\right\}.$$

Theorem 4.2. *With V_n^e defined as above (4.1), $V_n^e \otimes V_n^e$ is a finite dimensional representation of $\mathcal{R}_{1/2n, \kappa/2n}^e(\lambda) \in \mathcal{A}$, on which $\mathcal{R}_{1/2n, \kappa/2n}^e(\lambda)$ is equivalent to a Belavin R -matrix. The matrix coefficients of $\mathcal{R}_{1/2n, \kappa/2n}^e(\lambda)|_{V_n^e \otimes V_n^e}$ with respect to the basis $\{\psi_a \otimes \psi_b\}$ are given by*

$$R(\lambda)_{ij}^{kl} = \delta_{i+j, k+l(n)} \frac{\theta'_{\frac{1}{2}, \frac{1}{2}}(0, n\tau) \theta_{\frac{i-j}{n} + \frac{1}{2}, \frac{1}{2}}(\lambda - \kappa, n\tau)}{\theta_{\frac{i-k}{n} + \frac{1}{2}, \frac{1}{2}}(-\kappa, n\tau) \theta_{\frac{k-j}{n} + \frac{1}{2}, \frac{1}{2}}(\lambda, n\tau)}.$$

Proof. Set $\theta(z) = \vartheta_1(z, \tau) = -\theta_{\frac{1}{2}, \frac{1}{2}}(z, \tau)$, and $\mathcal{R}(\lambda) = \theta(\lambda) \mathcal{R}_{1/2n, \kappa/2n}^e(\lambda)$:

$$\begin{aligned} \mathcal{R}(\lambda) &= \theta(\lambda) \mathcal{F}_{1/2n, \kappa/2n}(-\lambda) \mathcal{R}^e(\lambda) \mathcal{F}_{1/2n, \kappa/2n}(\lambda) \\ &= \theta(\lambda) \left[G_e(z_1 - z_2 - \frac{\lambda + \kappa}{n}, \lambda) \tilde{\mathcal{F}}_{-\lambda/n} P - G_e(z_1 - z_2 - \frac{\lambda + \kappa}{n}, \kappa) \tilde{\mathcal{F}}_{-\kappa/n} \right] \\ &= \frac{\theta'(0) \theta(z_1 - z_2 - \frac{\lambda + \kappa}{n} + \lambda)}{\theta(z_1 - z_2 - \frac{\lambda + \kappa}{n})} \tilde{\mathcal{F}}_{-\lambda/n} P - \frac{\theta(\lambda) \theta'(0) \theta(z_1 - z_2 - \frac{\lambda + \kappa}{n} + \kappa)}{\theta(z_1 - z_2 - \frac{\lambda + \kappa}{n}) \theta(\kappa)} \tilde{\mathcal{F}}_{-\kappa/n}. \end{aligned}$$

Then \mathcal{R} an entire function of λ and $\mathcal{R}(0) = \theta'(0)P$. To see that $V_n^e \otimes V_n^e$ is invariant under \mathcal{R} , from the quasi-double periodicity of $\vartheta_1(z, \tau)$ we have:

$$\begin{aligned}\mathcal{R}(\lambda) \cdot f(z_1 + 1, z_2) &= (-1)^{n-1} \mathcal{R}(\lambda) \cdot f(z_1, z_2) = \mathcal{R}(\lambda) \cdot f(z_1, z_2 + 1) \\ \mathcal{R}(\lambda) \cdot f(z_1 + \tau, z_2) &= e^{-\pi i n \tau - 2\pi i n z_1} \mathcal{R}(\lambda) \cdot f(z_1, z_2), \text{ and} \\ \mathcal{R}(\lambda) \cdot f(z_1, z_2 + \tau) &= e^{-\pi i n \tau - 2\pi i n z_2} \mathcal{R}(\lambda) \cdot f(z_1, z_2)\end{aligned}$$

For the holomorphicity of $\mathcal{R}(\lambda) \cdot f$ in each variable, the possible poles of $\mathcal{R}(\lambda) \cdot f(z_1, z_2)$ occur where $z_1 - z_2 - \frac{\lambda + \kappa}{n} \equiv 0$, modulo the lattice Λ_τ . Observing that these simple zeros of the denominator are also zeros of $\vartheta_1(z_1 - z_2 - \frac{\lambda + \kappa}{n}) \mathcal{R}(\lambda) \cdot f(z_1, z_2)$, we see that $\mathcal{R}(\lambda) \cdot f(-, z_2), \mathcal{R}(\lambda) \cdot f(z_1, -) \in \mathcal{H}(\mathbb{C})$, and thus, $\mathcal{R}(\lambda)$ preserves $V_n^e \otimes V_n^e$. Henceforth, we denote the restriction of $\mathcal{R}(\lambda)$ to $V_n^e \otimes V_n^e$ by $R(\lambda)$.

To show that $R(\lambda)$ is equivalent to Belavin's R -matrix [2], $R_B(\lambda)$, define $S, T \in \text{End}(\mathcal{M}(\mathbb{C}))$ by:

$$(4.2) \quad \begin{aligned}(S \cdot f)(z) &= e^{\pi i(n-1)/n} f(z + \frac{1}{n}) \\ (T \cdot f)(z) &= e^{\pi i \tau/n - 2\pi i z} f(z - \frac{\tau}{n}).\end{aligned}$$

Note that V_n^e is invariant under S and T , and on V_n^e , $S^n = T^n = I$ and $TS = \omega ST$, where $\omega = e^{2\pi i/n}$.

The action of $S \otimes S$ and $T \otimes T$ on $V_n^e \otimes V_n^e$ is given by:

$$S \otimes S \cdot f(z_1, z_2) = e^{2\pi i(n-1)/n} f(z_1 + \frac{1}{n}, z_2 + \frac{1}{n}) = \omega^{-1} f(z_1 + \frac{1}{n}, z_2 + \frac{1}{n})$$

and

$$T \otimes T \cdot f(z_1, z_2) = e^{2\pi i \tau/n - 2\pi i(z_1 + z_2)} f(z_1 - \frac{\tau}{n}, z_2 - \frac{\tau}{n}),$$

and we obtain:

(a) $R(\lambda)$ is completely \mathbb{Z}_n symmetric, i.e.

$$\begin{aligned}(S \otimes S)^{-1} R(\lambda) (S \otimes S) &= R(\lambda) \\ (T \otimes T)^{-1} R(\lambda) (T \otimes T) &= R(\lambda)\end{aligned}$$

and

(b) $R(\lambda)$ has the following quasi-double periodicity:

$$\begin{aligned}(\text{b1}) \quad R(\lambda + 1) &= -(S \otimes 1)^{-1} R(\lambda) (S \otimes 1), \\ (\text{b2}) \quad R(\lambda + \tau) &= e^{-2\pi i(\xi + \lambda)} (T \otimes 1) R(\lambda) (T \otimes 1)^{-1}, \text{ with } \xi = -\frac{\kappa}{n} + \frac{\tau}{2} + \frac{1}{2}.\end{aligned}$$

To compute the matrix coefficients, $R(\lambda)_{ij}^{kl}$, with respect to the basis \mathcal{B} of Proposition 4.1, S and T act on \mathcal{B} by:

$$\begin{aligned}S \cdot \psi_a(z) &= \omega^a \psi_a(z), \\ T \cdot \psi_a(z) &= \psi_{a-1}(z).\end{aligned}$$

Hence, R is given in Belavin's representation by:

$$R(\lambda) = \sum_{\alpha \in \mathbb{Z}_n^{\times 2}} w_\alpha(\lambda) I_\alpha \otimes I_\alpha^{-1}$$

where $I_\alpha = S^{\alpha_1} T^{\alpha_2}$ and $w_\alpha \in \mathcal{H}(\mathbb{C})$. Thus,

$$R(\lambda)_{ij}^{kl} = \delta_{i+j, k+l(n)} \sum_{\gamma=0}^{n-1} w_{\gamma, i-k}(\lambda) \omega^{\gamma(k-j)}.$$

From (b1) and (b2), $w_\alpha(\lambda)$ has quasi-double periodicity:

$$\begin{aligned} w_\alpha(\lambda + 1) &= -\omega^{\alpha_2} w_\alpha(\lambda), \\ w_\alpha(\lambda + \tau) &= e^{-2\pi i(\xi + \lambda)} \omega^{\alpha_1} w_\alpha(\lambda), \quad \xi = -\frac{\kappa}{n} + \frac{\tau}{2} + \frac{1}{2} \end{aligned}$$

which, together with the initial condition $R(0) = \theta'(0)P$, uniquely determine

$$w_\alpha(\lambda) = w_{\alpha_1, \alpha_2}(\lambda) = \frac{\theta'(0)\theta_{\frac{1}{2} + \frac{\alpha_2}{n}, \frac{1}{2} - \frac{\alpha_1}{n}}(\lambda - \frac{\kappa}{n}, \tau)}{n\theta_{\frac{1}{2} + \frac{\alpha_2}{n}, \frac{1}{2} - \frac{\alpha_1}{n}}(-\frac{\kappa}{n}, \tau)}$$

and

$$R(\lambda)_{ij}^{kl} = \delta_{i+j, k+l(n)} \frac{\theta'_{\frac{1}{2}, \frac{1}{2}}(0, n\tau)\theta_{\frac{i-j}{n} + \frac{1}{2}, \frac{1}{2}}(\lambda - \kappa, n\tau)}{\theta_{\frac{i-k}{n} + \frac{1}{2}, \frac{1}{2}}(-\kappa, n\tau)\theta_{\frac{k-j}{n} + \frac{1}{2}, \frac{1}{2}}(\lambda, n\tau)}.$$

□

Remark 2. For n odd, these are the same representation spaces found by Felder and Pasquier to realize Belavin's R -matrices. Our choice of representation spaces for n even provide a concrete realization of the degeneration (given in the next section) from the Belavin R -matrices to the (twisted) trigonometric Cremmer-Gervais R -matrices.

At the trigonometric level, the twisted Shibukawa-Ueno R -operators $\mathcal{R}_{0, \kappa/2n}^t(\lambda)$ give rise to the affinized Cremmer-Gervais R -matrices (see Section 4.2). However, at the elliptic level, there is no such analog.

Theorem 4.3. *There are no finite dimensional subspaces of $\mathcal{M}(\mathbb{C}^2)$ invariant under $\mathcal{R}_{0, \beta}^e(\lambda)$.*

Proof. It suffices to show that $\mathcal{R}^e(\lambda)$ has no finite dimensional invariant subspaces. Suppose ρ is an eigenvalue of $\mathcal{R}^e(\lambda) = G(z_1 - z_2, \lambda)P - G(z_1 - z_2, \kappa)I$. Then

$$\frac{f(z_1, z_2)}{f(z_2, z_1)} = \frac{G(z_1 - z_2, \lambda)}{G(z_1 - z_2, \kappa) + \rho}.$$

Hence, $g(z_1, z_2) = \frac{G(z_1 - z_2, \lambda)}{G(z_1 - z_2, \kappa) + \rho}$ satisfies $g(z_2, z_1) = g(z_1, z_2)^{-1}$, or,

$$(4.3) \quad \rho^2 + [G(z, \kappa) + G(-z, \kappa)]\rho + G(z, \kappa)G(-z, \kappa) - G(z, \lambda)G(-z, \lambda) = 0.$$

Setting $x = 0$ in the 3-term equation,

$$\theta(y)^2\theta(z+w)\theta(z-w) + \theta(z)^2\theta(w+y)\theta(w-y) + \theta(w)^2\theta(y+z)\theta(y-z) = 0,$$

and we obtain

$$\begin{aligned} &G(z, \lambda)G(-z, \lambda) - G(z, \kappa)G(-z, \kappa) \\ &= \frac{\theta'(0)^2}{\theta(z)^2\theta(\lambda)^2\theta(\kappa)^2} [\theta(\kappa)^2\theta(z+\lambda)\theta(z-\lambda) - \theta(\lambda)^2\theta(z+\kappa)\theta(z-\kappa)] \\ &= \frac{\theta'(0)^2}{\theta(z)^2\theta(\lambda)^2\theta(\kappa)^2} [\theta(z)^2\theta(\kappa+\lambda)\theta(\kappa-\lambda)] \\ &= G(\kappa, \lambda)G(-\kappa, \lambda). \end{aligned}$$

That is, the constant term of (4.3) is z -independent.

The linear term of (4.3), $G(z, \kappa) + G(-z, \kappa) = \frac{\theta'(0)}{\theta(\kappa)} \left[\frac{\theta(z+\kappa) + \theta(z-\kappa)}{\theta(z)} \right]$ can be seen to be z -dependent by observing that $\theta(z+\kappa) + \theta(z-\kappa)$ is not a scalar multiple of $\theta(z)$, since it does not have the required quasi-periodicity with respect to τ . Hence, $\mathcal{R}^e(\lambda)$ has no eigenfunctions, and therefore has no finite dimensional invariant subspaces. \square

4.2. Trigonometric degeneration. For $k = 0, \dots, n-1$, consider the basis of V_n^e defined by

$$\begin{aligned} \tilde{\psi}_k(z) &= e^{-\pi i(k - \frac{n-1}{2})^2 \tau/n} \psi_k(z) \\ &= \sum_{m \equiv k(n)} e^{\pi i[(m - \frac{n-1}{2})^2 - (k - \frac{n-1}{2})^2] \tau/n} e^{2\pi i(m - \frac{n-1}{2})z}. \end{aligned}$$

For each k , set

$$(4.4) \quad \phi_k(z) \stackrel{\text{def}}{=} \lim_{\text{Im}\tau \rightarrow \infty} \tilde{\psi}_k(z) = e^{2\pi i(k - \frac{n-1}{2})z}$$

and define

$$(4.5) \quad V_n^t = \text{Span}\{\phi_k(z) \mid k = 0, \dots, n-1\} = \text{Span}\{e^{2\pi i l z} \mid l = -j, -j+1, \dots, j\}$$

where $j = \frac{n-1}{2} \in \frac{1}{2}\mathbb{Z}$.

In contrast to the elliptic case the complete 2-parameter family of twisted SU operators restrict simultaneously to this finite dimensional subspace.

Theorem 4.4. *The 2-parameter twisted trigonometric Shibukawa-Ueno operators $\mathcal{R}_{\alpha, \beta}^t(\lambda)$ restrict to $V_n^t \otimes V_n^t$, yielding a finite dimensional representation which is homogeneous with respect to the basis above. These representations are precisely the 2-parameter affinized Cremmer-Gervais operators described above (3.1).*

Proof. It was observed in [11] that $\mathcal{R}^t(\lambda)$ restricts to $V \otimes V$ where $V = \text{Span}\{e^{2\pi i k z}\}_{k=0}^{n-1}$. Since $V_n^t = e^{-\pi i(n-1)z} V$ and $\mathcal{R}^t(\lambda)$ commutes with the action of (multiplication by) any symmetric function (in particular, $e^{-\pi i(n-1)(z_1+z_2)}$), we see that $\mathcal{R}^t(\lambda)$ restricts to $V_n^t \otimes V_n^t$. Since $V_n^t \otimes V_n^t$ is also invariant under $\mathcal{F}_{\alpha, \beta}(\lambda)$, it follows that $V_n^t \otimes V_n^t$ is invariant under $\mathcal{R}_{\alpha, \beta}^t(\lambda)$.

For the matrix coefficients, with $\theta(z) = \sin \pi z$,

$$\frac{1}{2\pi i} \mathcal{R}_{\alpha, \beta}^t(\lambda) = \frac{\eta \zeta^{-2} w_1 - \eta^{-1} \gamma^2 w_2}{\hat{\eta}(\zeta^{-2} w_1 - \gamma^2 w_2)} \tilde{\mathcal{F}}_{\zeta^2} P - \frac{q \zeta^{-2} w_1 - q^{-1} \gamma^2 w_2}{\hat{q}(\zeta^{-2} w_1 - \gamma^2 w_2)} \tilde{\mathcal{F}}_{\gamma^2}$$

where $w_k = e^{2\pi i z_k}$, $\gamma = e^{2\pi i \beta}$, $q = e^{\pi i \kappa}$, $\eta = e^{\pi i \lambda}$, $\zeta = e^{2\pi i \alpha \lambda}$, and $\tilde{\mathcal{F}}_s \cdot f(w_1, w_2) = f(s^{-1} w_1, s w_2)$.

With respect to the basis $\{\phi_a(z_1) \phi_b(z_2)\}$ defined by (4.4), the matrix coefficients, given by $R_{\alpha, \beta}^t(\lambda) \cdot \phi_i(z_1) \phi_j(z_2) = \sum_{k, l} R_{\alpha, \beta}^t(\lambda)_{ij}^{kl} \phi_k(z_1) \phi_l(z_2)$, can be found by direct calculation, or by earlier observations regarding homogeneous operators. The R -matrices $R^t(\lambda)$ (whose matrix coefficients are given in [11]) are homogeneous, and the operator $F_{\alpha, \beta}(\lambda)$ acts diagonally on V_n^t so that $F_{\alpha, \beta}(\lambda) \cdot \phi_a(z_1) \phi_b(z_2) = \exp[2\pi i(a -$

b) $(\alpha\lambda - \beta)]\phi_a(z_1)\phi_b(z_2) = \zeta^{a-b}\gamma^{b-a}\phi_a(z_1)\phi_b(z_2)$. Thus, we obtain

$$R_{\alpha,\beta}^t(\lambda)_{ij}^{kl} = \zeta^{2(i-k)}\gamma^{2(j-k)}R^t(\lambda)_{ij}^{kl}$$

$$= \frac{2\pi i}{\hat{q}\hat{\eta}}\zeta^{2(i-k)}\gamma^{2(j-k)} \begin{cases} q\eta^{-1} & i = j = k = l, \\ -\hat{\eta}q^{\text{sgn}(i-j)} & i = k \neq j = l, \\ \hat{q}\eta^{\text{sgn}(j-i)} & l = i \neq k = j, \\ \text{sgn}(j-i)\hat{q}\hat{\eta} & \min(i,j) < k < \max(i,j) \\ & \text{and } i+j = k+l, \\ 0 & \text{otherwise.} \end{cases}$$

When $\alpha = 0$ and $\beta = \kappa/2n$ (so $\gamma^n = q$), these are the standard affinizations of the Cremmer-Gervais R -matrices. More generally, we have (up to scalar multiple) $R_{\alpha,\beta}^t(\lambda) = R_{CG(\alpha,\beta-\kappa/2n)}(\lambda)$. \square

This yields a natural degeneration of the Belavin R -matrix, $R_B(\lambda) = R_{1/2n,\kappa/2n}^e(\lambda)$, into a certain kind of twisted affinized Cremmer-Gervais matrix, namely $R_{CG(1/2n,0)}(\lambda)$. Set $R_{1/2n,\kappa/2n}^\theta = R_{\kappa}^\theta$. From (4.4), together with

$$\lim_{\text{Im } \tau \rightarrow \infty} R_{\kappa}^e(\lambda)\tilde{\psi}_a(z_1)\tilde{\psi}_b(z_2) = R_{\kappa}^t(\lambda)\phi_a(z_1)\phi_b(z_2),$$

we obtain

Theorem 4.5. *The matrices $R_{CG(1/2n,0)}(\lambda)$ are trigonometric degenerations of Belavin's R -matrices. The degeneration is given by*

$$\lim_{\text{Im } \tau \rightarrow \infty} (G \otimes G)^{-1}R_B(\lambda)(G \otimes G) = R_{CG(1/2n,0)}(\lambda)$$

where G is defined by $G_{ab} = \delta_{ab}e^{-\pi i(a-\frac{n-1}{2})^2\tau/n}$ for $a, b = 0, \dots, n-1$.

Remark 3. This result is analogous to one obtained by Antonov, Hasegawa, and Zabrodin [1].

4.3. Rational degeneration. The trigonometric to rational degeneration is much more straightforward than the elliptic to trigonometric degeneration. We omit the details which are easy to verify. Essentially we find that the degeneration of the Shibukawa-Ueno operators induces, via an appropriate representation, an affinized version of the degeneration of the Cremmer-Gervais R -matrices into the "Jordan-Cremmer-Gervais" operators described in [5].

Let us first look at twisted rational Shibukawa-Ueno operators and their representations. In the rational case, when $\theta(z) = z$, the Shibukawa-Ueno operators have the simple form:

$$\mathcal{R}^r(\lambda) \cdot f(z_1, z_2) = \frac{z_1 - z_2 + \lambda}{(z_1 - z_2)\lambda} f(z_2, z_1) - \frac{z_1 - z_2 + \kappa}{(z_1 - z_2)\kappa} f(z_1, z_2)$$

while the twisted versions have the form

$$\mathcal{R}_{\alpha,\beta}^r(\lambda) \cdot f(z_1, z_2) = \frac{z_1 - z_2 - 2(\alpha\lambda + \beta) + \lambda}{(z_1 - z_2 - 2(\alpha\lambda + \beta))\lambda} f(z_2 + 2\alpha\lambda, z_1 - 2\alpha\lambda) \\ - \frac{z_1 - z_2 - 2(\alpha\lambda + \beta) + \kappa}{(z_1 - z_2 - 2(\alpha\lambda + \beta))\kappa} f(z_1 - 2\beta, z_2 + 2\beta)$$

Let $V_n^r = \text{Span}\{z^k \mid k = 0, \dots, n-1\}$. It is easily seen that the above operators restrict to $V_n^r \otimes V_n^r$. Moreover $V_n^r \otimes V_n^r$ is invariant under $\mathcal{F}_{\alpha,\beta}(\lambda)$, so that the

restriction to these finite dimensional spaces commutes with the twisting operation. Fixing n , denote by $R^r(\lambda)$ and $R_{\alpha,\beta}^r(\lambda)$, respectively, the restrictions of $\mathcal{R}^r(\lambda)$ and $\mathcal{R}_{\alpha,\beta}^r(\lambda)$ to $V_n^r \otimes V_n^r$. Looking explicitly at the case $\alpha = 0$, we see that

$$R_{0,\beta}^r(\lambda) \cdot z_1^i z_2^j = \frac{1}{\lambda} z_2^i z_1^j - \frac{1}{\kappa} (z_1 - 2\beta)^i (z_2 + 2\beta)^j + \frac{z_2^i z_1^j - (z_1 - 2\beta)^i (z_2 + 2\beta)^j}{(z_1 - z_2 - 2\beta)}.$$

Set $R_\beta^r = \lim_{\lambda \rightarrow \infty} R_{0,\beta}^r(\lambda)$. Then R_β^r is the general Jordan-Cremmer-Gervais operator introduced in [5] and $R_{0,\beta}^r(\lambda)$ is an affinization of this constant operator, in the sense [10, page 296] that

$$R_{0,\beta}^r(\lambda) = (1/\lambda)P + R_\beta^r.$$

Thus, $R_{\alpha,\beta}^r(\lambda)$ are twists of these affinized Jordan-Cremmer-Gervais R -matrices, which we denote by $R_{JCG(\alpha,\beta)}(\lambda)$.

For the rational degeneration, we introduce the period $2\tau_1$, taking $\mathcal{R}^t(\lambda)$ to be the trigonometric Shibukawa-Ueno R -operator with $\theta(z) = \sin(\pi z/\tau_1)$, and $\phi_k(z, \tau_1) = \phi_k(z/\tau_1)$. Then $\mathcal{R}_{\alpha,\beta}^r(\lambda) = \lim_{\tau_1 \rightarrow \infty} \mathcal{R}_{\alpha,\beta}^t(\lambda)$.

As in the previous section, the vector space $V_n^t(\tau_1) = \text{Span}\{\phi_k(z, \tau_1)\}$ yields a representation of $\mathcal{R}^t(\lambda)$. Consider the basis of $V_n^t(\tau_1)$ defined by

$$(4.6) \quad \tilde{\phi}_k(z) = \tilde{\phi}_k(z, \tau_1) = \sum_{l=0}^k (-1)^{k-l} \left(\frac{\tau_1}{2\pi i}\right)^k \binom{k}{l} \phi_l(z, \tau_1).$$

Then for each k , $\lim_{\tau_1 \rightarrow \infty} \tilde{\phi}_k(z, \tau_1) = z^k$. This yields the following degeneration of the 2-parameter affinized Cremmer-Gervais R -matrices into this 2-parameter family of Jordan-Cremmer-Gervais R -matrices:

Theorem 4.6. *The representation of $\mathcal{R}_{\alpha,\beta}^t$ on $V_n^t \otimes V_n^t$ degenerates as $\tau_1 \rightarrow \infty$ into the representation of $\mathcal{R}_{\alpha,\beta}^r$ on $V_n^r \otimes V_n^r$. This yields a degeneration of the matrix $R_{JCG(\alpha,\beta)}$ into $R_{JCG(\alpha,\beta)}$ given by*

$$\lim_{\tau_1 \rightarrow \infty} (H \otimes H)^{-1} R_{JCG(\alpha,\beta)}(\lambda) (H \otimes H) = R_{JCG(\alpha,\beta)}(\lambda),$$

where H is defined by

$$H_{ab} = \left(\frac{\tau_1}{2\pi i}\right)^b (-1)^{b-a} \binom{a}{b}, \quad a, b = 0, \dots, n-1.$$

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UNIVERSITY OF CALIFORNIA AT DAVIS, DAVIS, CA 95616, U.S.A.
E-mail address: `endelman@math.ucdavis.edu`

UNIVERSITY OF CINCINNATI, CINCINNATI, OH 45221-0025
E-mail address: `timothy.hodges@uc.edu`