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Generalized Jordanian *R*-Matrices of Cremmer–Gervais Type

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Abstract. An explicit quantization is given of certain skew-symmetric solutions of the classical Yang–Baxter equation, yielding a family of *R*-matrices which generalize to higher dimensions the Jordanian *R*-matrices. Three different approaches to their construction are given: as twists of degenerations of the Shibukawa–Ueno, Yang–Baxter operators on meromorphic functions; as boundary solutions of the quantum Yang–Baxter equation; via a vertex-IRF transformation from solutions to the dynamical Yang–Baxter equation.

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Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. The skew-symmetric solutions of the classical Yang–Baxter equation for a simple Lie algebra are classified by the quasi-Frobenius subalgebras; that is, pairs of the form $(\mathfrak{f}, \omega)$ where $\mathfrak{f}$ is a subalgebra and $\omega\colon \mathfrak{f} \wedge \mathfrak{f} \to \mathbb{F}$ is a nondegenerate 2-cocycle on $\mathfrak{f}$. By a result of Drinfeld [6], the associated Lie bialgebras admit quantizations. This is done by twisting the enveloping algebra $U(\mathfrak{g})[[h]]$ by an appropriate Hopf algebra 2-cocycle. However, neither construction lends itself easily to direct calculation and few explicit examples exist to illustrate this theory. The most well-known is the Jordanian quantum group [4,16] associated to the classical r-matrix $E \wedge H$ inside $\mathfrak{sl}(2) \otimes \mathfrak{sl}(2)$. In [12], Gerstenhaber and Giaquinto constructed explicitly the r-matrix $r_{\mathfrak{p}}$ associated to certain maximal parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{sl}(n)$. In particular for the parabolic subalgebra $\mathfrak{p}$ generated by $\mathfrak{b}^+$ and $F_1, \dots F_{n-2}$, their construction yields

$$r_{\mathfrak{p}} = n \sum_{i<j} \sum_{k=i}^{j-1} E_{k,i} \wedge E_{i+j-k-1,j} + \sum_{i,j} (j-1) E_{j-1,j} \wedge E_{i,i}$$

In [13], they raise the problem of quantizing this r-matrix, in the sense of constructing

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an invertible $R \in M_n(\mathbb{F}) \otimes M_n(\mathbb{F}) \otimes \mathbb{F}[[h]]$ satisfying the Yang–Baxter equation and of the form $I + hr + O(h^2)$. When $n = 2$, the solution is the well-known Jordanian R-matrix. Gerstenhaber and Giaquinto construct a quantization of $r_{\mathfrak{p}}$ in the $n = 3$ case and verify the necessary relations by direct calculation. We give below the quantization of $r_{\mathfrak{p}}$ in the general case. Moreover, we are able to give three separate constructions which emphasize the fundamental position occupied by this R-matrix.

In the first section we construct R (somewhat indirectly) as an extreme degeneration of the Belavin R-matrix. We do this by following the construction by Shibukawa and Ueno of solutions of the Yang–Baxter equation for linear operators on meromorphic functions. In [17], they showed that from any solution of Riemann's three-term equation, they could construct such a solution of the Yang–Baxter equation. These solutions occur in three types: elliptic, trigonometric and rational. Felder and Pasquier [11] showed that in the elliptic case, these operators, after twisting and restricting to suitable finite dimensional subspaces, yield Belavin's R-matrices. In the trigonometric case, the same procedure yields the affinization of the Cremmer–Gervais quantum groups; sending the spectral parameter to infinity then yields the Cremmer–Gervais R-matrices themselves. Repeating this procedure in the rational case yields the desired quantization of $r_{\mathfrak{p}}$, which we shall denote $R_{\mathfrak{p}}$.

In the second section we show that these R-matrices occur as boundary solutions of the modified quantum Yang–Baxter equation, in the sense of Gerstenhaber and Giaquinto [13]. It was observed in [12] that if $\mathfrak{M}$ is the set of solutions of the modified classical Yang–Baxter equation, then $\mathfrak{M}$ is a locally closed subset of $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g})$ and $\bar{\mathfrak{M}} - \mathfrak{M}$ consists of solutions to the classical Yang–Baxter equation. The element $r_{\mathfrak{p}}$ was found to lie on the boundary of the orbit under the adjoint action of $\mathrm{SL}(n)$ of the modified Cremmer–Gervais r-matrix. In [13], Gerstenhaber and Giaquinto began an investigation into the analogous notion of boundary solutions of the quantum Yang–Baxter equation. They conjectured that the boundary solutions to the classical Yang–Baxter equation described above should admit quantizations which would be on the boundary of the solutions of their modified quantum Yang–Baxter equation. They confirmed this conjecture for the Cremmer–Gervais r-matrix in the $\mathfrak{sl}(3)$ case using some explicit calculations. We prove the conjecture for the general Cremmer–Gervais r-matrix by verifying that the matrices $R_{\mathfrak{p}}$ do indeed lie on the boundary of the set of solutions to the modified quantum Yang–Baxter equation.

In the third section we show that these matrices may also be constructed via a 'Vertex-IRF' transformation from certain solutions of the dynamical Yang–Baxter equation given in [7]. This construction is analogous to the original construction of the Cremmer–Gervais R-matrices given in [3].

The position of R_{p} with relation to other fundamental solutions of the YBE and DYBE can be summarized heuristically by the diagram below.

$$\begin{tikzcd}
R_B \arrow[d] & & & R_F \arrow[d] & \\
\hat{R}_{CG} \arrow[r] \arrow[d] & R_{CG} \arrow[d] & & \hat{R}_{GN} \arrow[r] \arrow[d] & R_{GN} \arrow[d] \\
R_{B,r} \arrow[r] & R_{\mathrm{p}} & & R_{F,r} \arrow[r] & R_{GN,r} \\
\text{YBE} & & & \text{DYBE} &
\end{tikzcd}$$

On the left-hand side, R_B is Belavin's elliptic R-matrix; R_{CG} the Cremmer–Gervais R-matrix; $\hat{R}_{CG}$ is the affinization of R_{CG} which is also the trigonometric degeneration of the Belavin R-matrix; $R_{B,r}$ is a rational degeneration of the Belavin R-matrix. The vertical arrows denote degeneration of the coefficient functions (from elliptic to trigonometric and from trigonometric to linear); the horizontal arrows denote the limit as the spectral parameter tends to infinity. On the right-hand side, R_F is Felder's elliptic dynamical R-matrix; $\hat{R}_{GN}$ and $R_{F,r}$ are trigonometric and rational degenerations; R_{GN} is the Gervais–Neveu dynamical R-matrix and $R_{GN,r}$ is a rational degeneration of the Gervais–Neveu matrix given in [7]. The passage between the two diagrams is performed by Vertex-IRF transformations. The relationships involved in the top two lines of this diagram are well known [1, 3, 10]. This paper is concerned with elucidating the position of R_{p} in this picture.

1. Construction of R_{p}

1.1. THE YBE FOR OPERATORS ON FUNCTION FIELDS

Recall that if A is an integral domain and σ is an automorphism of A, then σ extends naturally to the field of rational functions $A(x)$ by acting on the coefficients. Denote by $\mathbb{F}(z_1, z_2)$ the field of rational functions in the variables z_1 and z_2. Then for any $\sigma \in \mathrm{Aut}\mathbb{F}(z_1, z_2)$, and any $i, j \in \{1, 2, 3\}$, we may define $\sigma_{ij} \in \mathrm{Aut}\mathbb{F}(z_1, z_2, z_3)$ by realizing $\mathbb{F}(z_1, z_2, z_3)$ as $\mathbb{F}(z_i, z_j)(z_k)$. Set $\Gamma = \mathrm{Aut}\mathbb{F}(z_1, z_2)$. Elements $R = \sum \alpha_i(z_1, z_2)\sigma_i$ of the group algebra $\mathbb{F}(z_1, z_2)[\Gamma]$ act as linear operators on $\mathbb{F}(z_1, z_2)$ and we may define in this way R_{ij} as linear operators on $\mathbb{F}(z_1, z_2, z_3)$. Thus we may look for solutions of the Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ amongst such operators. Denote by P the operator $P \cdot f(z_1, z_2) = f(z_2, z_1)$.

THEOREM 1.1. *The operator*

$$R = -\frac{\kappa}{z_1 - z_2}P + \left(1 + \frac{\kappa}{z_1 - z_2}\right)I = I + \frac{\kappa}{z_1 - z_2}(I - P)$$

satisfies the Yang–Baxter equation for any $\kappa \in \mathbb{F}$.

Proof. Consider an operator of the general form

$$R = \alpha(z_1 - z_2)P + \beta(z_1 - z_2)I.$$

Then it is easily seen that R satisfies the Yang–Baxter equation if and only if

$$\alpha(x)\alpha(y) = \alpha(x - y)\alpha(y) + \alpha(x)\alpha(y - x)$$

and

$$\alpha(x)\alpha(y)^2 + \beta(y)\beta(-y)\alpha(x + y) = \alpha(x)^2\alpha(y) + \beta(x)\beta(-x)\alpha(x + y).$$

These equations are satisfied when $\alpha(x) = -\kappa/x$ and $\beta(x) = 1 - \alpha(x)$. Moreover these are essentially the only such solutions [5]. □

In fact, (at least when $\mathbb{F}$ is the field of complex numbers) this operator is the limit as the spectral parameter tends to infinity of certain solutions of the Yang–Baxter equation with spectral parameter on meromorphic functions constructed by Shibukawa and Ueno. Recall that in [17], they showed that operators of the form

$$R(\lambda) = G(z_1 - z_2, \lambda)P - G(z_1 - z_2, \kappa)I$$

satisfied the Yang–Baxter equation

$$R_{12}(\lambda_1)R_{13}(\lambda_1 + \lambda_2)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1 + \lambda_2)R_{12}(\lambda_1)$$

for any $\kappa \in \mathbb{F}$ if G was of the form

$$G(z, \lambda) = \frac{\theta'(0)\theta(\lambda + z)}{\theta(\lambda)\theta(z)}$$

and θ satisfied the equation

$$\theta(x + y)\theta(x - y)\theta(z + w)\theta(z - w) + \theta(x + z)\theta(x - z)\theta(w + y)\theta(w - y) +$$

$$+\theta(x + w)\theta(x - w)\theta(y + z)\theta(y - z) = 0$$

The principal solution of this equation is $\theta(z) = \theta_1(z)$, the usual theta function (as defined in, say, [18]), along with the degenerations of the theta functions, $\sin(z)$ and z, as one or both of the periods tend to infinity. Felder and Pasquier [11] showed that in the case where θ is a true theta function, these operators, when twisted and restricted to suitable subspaces, yield the Belavin R-matrices. When θ is trigonometric, the operator yields in a similar way the affinizations of the Cremmer–Gervais R-matrices [5]. Letting the spectral parameter tend to infinity in a suitable way yields a constant solution of the YBE on the function field which again yields the usual Cremmer–Gervais R-matrices on restriction to finite-dimensional subspaces. In the rational case, the same twisting and restriction procedure yields the desired quantization of r_p.

When $\theta(z) = z$ we have $G(z, \lambda) = 1/\lambda + 1/z$. Sending λ to infinity (and adjusting by a factor of $-\kappa$), we obtain the solution of the Yang–Baxter equation given in the theorem above. Write $R = I + \kappa r$ where $r = (I - P)/(z_1 - z_2)$. Then r is a particularly interesting operator. It satisfies the classical Yang–Baxter equation, both forms of the quantum Yang–Baxter equation and has square zero. Its quantization is then just the exponential $\exp \kappa r = I + \kappa r = R$.

Let V_n be the space of polynomials in z_1 of degree less than n. Then we may identify the space $V_n \otimes V_n$ with the subspace of $\mathbb{F}(z_1, z_2)$ consisting of polynomials of degree less than n in both z_1 and z_2. Since $R \cdot z_1^i z_2^j = z_1^i z_2^j + \kappa(z_1^i z_2^j - z_2^i z_1^j)/(z_1 - z_2)$, R restricts to an operator on $V_n \otimes V_n$. With respect to the natural basis, R has the form

$$R(e_i \otimes e_j) = e_i \otimes e_j - \kappa \sum_k \eta(i,j,k) e_k \otimes e_{i+j-k-1},$$

where

$$\eta(i,j,k) = \begin{cases} 1 & \text{if } i \leqslant k < j, \\ -1 & \text{if } j \leqslant k < i, \\ 0 & \text{otherwise .} \end{cases}$$

We now apply a simple twist. Define the operator $\tilde{F}_p$ by $\tilde{F}_p \cdot f(z_1, z_2) = f(z_1 + p, z_2 - p)$.

LEMMA 1.2. *Let $F = \tilde{F}_p$. Then F and the above R satisfy:*

(1) $F_{21} = F_{12}^{-1}$,
(2) $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$,
(3) $R_{12}F_{23}F_{13} = F_{13}F_{23}R_{12}$,
(4) $R_{23}F_{12}F_{13} = F_{13}F_{12}R_{23}$.

Hence, $R_F = F_{21}^{-1}RF_{12}$ also satisfies the Yang–Baxter equation.

Proof. The four relations are routine verifications. The fact that R_F then satisfies the Yang–Baxter equation is a well-known fact about R-matrices extended to this slightly more general situation. □

Notice that $F_{21}^{-1}PF_{12} = P$ and $F_{21}^{-1}F_{12} = F^2 = \tilde{F}_{2p}$. Taking $p = h/2$ yields

$$R_F = \tilde{F}_h + \frac{\kappa}{z_1 - z_2 + h}(\tilde{F}_h - P).$$

Notice that

$$R_F \cdot z_1^i z_2^j = (z_1 + h)^i (z_2 - h)^j + \kappa \frac{(z_1 + h)^i (z_2 - h)^j - z_2^i z_1^j}{z_1 - z_2 + h}$$

and again R_F restricts to an operator on $V_n \otimes V_n$.

DEFINITION 1.3. Let n be a positive integer. Define

$$R_{\mathrm{p}} = \tilde{F}_h - \frac{hn}{z_1 - z_2 + h}(\tilde{F}_h - P)$$

restricted to $V_n \otimes V_n$.

Putting all the above together yields the main result.

THEOREM 1.4.*For any $h \in \mathbb{F}$ and positive integer n, R_{p} satisfies the Yang–Baxter equation.*

1.2. EXPLICIT FORM OF R_{p}

We now find an explicit formula for the matrix coefficients of R_{p} with respect to the natural basis.

Define the coefficients of R_{p} by $R_{\mathrm{p}} \cdot z_1^i z_2^j = \sum_{a,b} R_{ij}^{ab} z_1^a z_2^b$.

PROPOSITION 1.5. *The coefficients of R_{p} are given by*

$$R_{ij}^{ab} = (-1)^{j-b}\left[\binom{i}{a}\binom{j}{b} + n\sum_k (-1)^{k-a}\binom{i}{k}\binom{j+k-a-1}{b}\eta(j,k,a)\right]h^{i+j-a-b}.$$

Proof. Recall that

$$R_{\mathrm{p}} \cdot z_1^i z_2^j = (z_1 + h)^i (z_2 - h)^j - hn\frac{(z_1 + h)^i (z_2 - h)^j - z_2^i z_1^j}{z_1 - z_2 + h}$$

For the second term we note that

$$\frac{z_1^j z_2^i - (z_1 + h)^i (z_2 - h)^j}{z_1 - z_2 + h}$$

$$= \sum_{k,b,a} (-1)^{j+k-a-b}\binom{i}{k}\binom{j+k-a-1}{b}\eta(j,k,a)h^{i+j-a-b-1} z_1^a z_2^b$$

Combining this with the binomial expansion of the first term yields the assertion. □

The explicit form of this matrix in the case when $n = 3$ can be found in [13, Page 136].

1.3. THE SEMICLASSICAL LIMIT

The operator R_{p} is a polynomial function of the parameter h of the form $I + rh + \mathrm{O}(h^2)$. By working over a suitably extended field, we may assume that h is a formal parameter. Hence, r satisfies the classical Yang–Baxter equation. We now verify that r is the boundary solution r_{p} associated to the classical Cremmer–Gervais r-matrix found by Gerstenhaber and Giaquinto in [12].

Recall that their solution of the CYBE on the boundary of the component containing the modified Cremmer–Gervais r-matrix was (up to a scalar)

$$b_{CG} = n\sum_{i<j}\sum_{k=1}^{j-i} E_{i,j-k+1}\wedge E_{j,i+k} + \sum_{i,j}(n-j)E_{i,i}\wedge E_{j,j+1}.$$

(Here as usual we are taking the E_{ij} to be the basis of $\operatorname{End}V$ defined by $E_{ij}e_k = \delta_{jk}e_i$ for a fixed basis $\{e_1,\ldots,e_n\}$ of V; we shall use the convention $x\wedge y = x\otimes y - y\otimes x$). To pass from the b_{CG} to our matrix r_{p}, one applies the automorphism $\phi(E_{ij}) = -E_{n+1-j,n+1-i}$. Thus our matrix is again a boundary solution but for a Cremmer–Gervais r-matrix associated to a different choice of parabolic subalgebras.

THEOREM 1.6. *The operator* R_{p} *is of the form* $I + r_{\mathrm{p}}h + \mathrm{O}(h^2)$, *where*

$$r_{\mathrm{p}}\cdot z_1^i z_2^j = n\sum \eta(i,j,k)z_1^k z_2^{i+j-k-1} + iz_1^{i-1}z_2^j - jz_1^i z_2^{j-1}.$$

In particular the matrix representation of r_{p} *with respect to the usual basis is*

$$n\sum_{i<j}\sum_{k=i}^{j-1} E_{k,i}\wedge E_{i+j-k-1,j} + \sum_{i,j}(j-1)E_{j-1,j}\wedge E_{i,i}.$$

Proof. From Proposition 1.5, the coefficients r_{ij}^{ab} are nonzero only when $b = i+j-a-1$ and in this case,

$$\begin{aligned} r_{ij}^{a,i+j-a-1} &= \frac{1}{h}R_{ij}^{a,i+j-a-1} = (-1)^{a-i+1}\binom{i}{a}\binom{j}{a-i+1} + n\eta(i,j,a) \\ &= i\delta_{a,i-1} - j\delta_{a,i} + n\eta(i,j,a). \end{aligned}$$

Hence

$$r_{\mathrm{p}}\cdot z_1^i z_2^j = n\sum \eta(i,j,k)z_1^k z_2^{i+j-k-1} + iz_1^{i-1}z_2^j - jz_1^i z_2^{j-1}.$$

Thus interpreting r_{p} as an operator on $V\otimes V$ we get

$$r_{\mathrm{p}}\cdot e_i\otimes e_j = n\sum \eta(i,j,k)e_k\otimes e_{i+j-k-1} + (i-1)e_{i-1}\otimes e_j - (j-1)e_i\otimes e_{j-1}.$$

In matrix form,

$$r_{\mathrm{p}} = n\sum_{i<j}\sum_{k=i}^{j-1} E_{k,i}\wedge E_{i+j-k-1,j} + \sum_{i,j}(j-1)E_{j-1,j}\wedge E_{i,i}.$$

□

2. Boundary Solutions of the Yang–Baxter Equation

2.1. THE MODIFIED YANG–BAXTER EQUATION

In [13], Gerstenhaber and Giaquinto introduced the *modified (quantum) Yang–Baxter equation* (MQYBE). An operator $R \in \operatorname{End} V \otimes V$ is said to satisfy the MQYBE if

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = \lambda(P_{123}R_{12} - P_{213}R_{23})$$

for some nonzero λ in $\mathbb{F}$. Here by P_{ijk} we mean the permutation operator $P_{ijk}(v_1 \otimes v_2 \otimes v_3) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$ where σ is the permutation (ijk).

Denote by $\mathfrak{R}$ the set of solutions of the YBE in $\operatorname{End} V \otimes V$ and by $\mathfrak{R}'$ the set of solutions of the MQYBE. Then $\mathfrak{R}'$ is a quasi-projective subvariety of $\mathbb{P}(M_{n^2}(\mathbb{F}))$ and $\bar{\mathfrak{R}}' - \mathfrak{R}'$ is contained in $\mathfrak{R}$ [13]. The elements of $\bar{\mathfrak{R}}' - \mathfrak{R}'$ are naturally called *boundary* solutions of the YBE. Little is currently known about this set though we conjecture that it contains some interesting R-matrices closely related to the quantizations of Belavin–Drinfeld r-matrices [9]. Let R be a solution of the YBE for which PR satisfies the Hecke equation $(PR - q)(PR + q^{-1}) = 0$. Set $\lambda = (1 - q^2)^2/(1 + q^2)^2$. Then $Q = (2R + (q^{-1} - q)P)/(q + q^{-1})$ is a unitary solution of the MQYBE. Roughly speaking what we expect to find is the following. If R is a quantization (in the algebraic sense) of a Belavin–Drinfeld r-matrix on $\mathfrak{sl}(n)$, then on the boundary of the component of $\mathfrak{R}'$ containing Q, we should find the quantization of the skew-symmetric r-matrix associated (in the sense of Stolin) with the parabolic subalgebra of $\mathfrak{sl}(n)$ associated to r. We prove this conjecture here for the most well-known example, the Cremmer–Gervais R-matrices.

If $R \in \operatorname{End}(V \otimes V)\hat{\otimes}\mathbb{F}[[h]]$ satisfies the QYBE and is of the form $I + hr + O(h^2)$, then r satisfies the classical Yang–Baxter equation and R is said to be a quantization of r. The situation for the MQYBE is slightly more complicated and applies only to the $\mathfrak{sl}(n)$ case. Recall that the modified classical Yang–Baxter equation (MCYBE) for an element $r \in \mathfrak{sl}(n) \otimes \mathfrak{sl}(n)$ is the equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \mu\Omega,$$

where Ω is the unique invariant element of $\wedge^3\mathfrak{sl}(n)$ (which in the standard representation is the operator $P_{123} - P_{213}$). If R is of the form $I + hr + O(h^2)$ and is a solution of the MQYBE then λ is of the form $vh^2 + O(h^3)$ for some scalar v. If $v \neq 0$, then r satisfies the MCYBE. In this case we say that R is a quantization of r.

There is an analogous notion of boundary solution for the classical Yang–Baxter equation. In [12], Gerstenhaber and Giaquinto showed that the matrix b_{CG} lies on the boundary of the component of the set of solutions to the MCYBE containing the modified Cremmer–Gervais classical r-matrix. They conjectured that its quantization should lie on the boundary of the component of $\mathfrak{R}'$ containing the modified Cremmer–Gervais R-matrix and proved this in the case $n = 3$ in [13]. We prove now this conjecture in general by showing that R_{p} lies on the boundary of this component of $\mathfrak{R}'$.

2.2. THE CREMMER–GERVAIS SOLUTION OF THE MQYBE

Consider the linear operator on $\mathbb{F}(z_1, z_2)$

$$R = \frac{\hat{q}pz_2}{pz_2 - z_1}P + \left(q - \frac{\hat{q}pz_2}{pz_2 - z_1}\right)F_p$$

where $\hat{q} = q - q^{-1}$ and $F_p \cdot f(z_1, z_2) = f(p^{-1}z_1, pz_2)$. When restricted to $V_n \otimes V_n$, the above operator becomes the usual 2-parameter Cremmer–Gervais R-matrix [5,14]. When $p^n = q^2$, this is the original Cremmer–Gervais R-matrix which induces a quantization of SL(n) [3].

If R is any solution of the YBE for which PR satisfies the Hecke equation $(PR - q)(PR + q^{-1}) = 0$ then $Q = (2R + (q^{-1} - q)P)/(q + q^{-1})$ is a unitary solution of the MQYBE for $\lambda = (1 - q^2)^2/(1 + q^2)^2$. Hence, the operator $Q_{p,q} = (2R - \hat{q}P)/(q + q^{-1})$ satisfies the MQYBE. Explicitly,

$$Q_{p,q} = F_p - \frac{\hat{q}(z_2 + p^{-1}z_1)}{(q + q^{-1})(z_2 - p^{-1}z_1)}(F_p - P).$$

We call the corresponding matrices induced from these operators, the modified Cremmer–Gervais R-matrices.

2.3. DEFORMATION TO THE BOUNDARY

Henceforth, take $q^2 = p^n$. Then the operator $Q_{p,q}$ becomes

$$Q_p = F_p - \frac{(p^n - 1)(z_2 + p^{-1}z_1)}{(p^n + 1)(z_2 - p^{-1}z_1)}(F_p - P).$$

This is the modified version of the one-parameter Cremmer–Gervais operator described above. Again Q_p may be restricted to the subspace $V_n \otimes V_n$ where its action is given by

$$Q_p \cdot z_1^i z_2^j = p^{j-i} z_1^i z_2^j - \frac{(p^n - 1)}{(p^n + 1)} \sum [\eta(i, j, k) + \eta(i, j, k - 1)] p^{j-k} z_1^k z_2^{i+j-k}.$$

Fix $h \in \mathbb{F}$ and $p \in \mathbb{F}^*$, define $\tilde{F}_{p,h}$ by $\tilde{F}_{p,h} \cdot f(z_1, z_2) = f(p^{-1}z_1 + p^{-1}h, pz_2 - h)$. Further, define

$$\begin{aligned} B_{p,h,n} = {} & \tilde{F}_{p,h} - \frac{(p^n - 1)(pz_2 + z_1)}{(p^n + 1)(pz_2 - z_1 - h)}(\tilde{F}_{p,h} - P) + \\ & + \frac{h(p^n - 1)(p + 1)}{(p^n + 1)(p - 1)(pz_2 - z_1 - h)}(\tilde{F}_{p,h} - P). \end{aligned}$$

Note that

$$B_{1,h,n} = \frac{hn}{(z_2 - z_1 - h)}(\tilde{F}_h - P) + \tilde{F}_h$$

since $\tilde{F}_h = \tilde{F}_{1,h}$. This is the operator R_F described above (with $\kappa = -hn$) that restricts to R_{p} on finite-dimensional subspaces.

PROPOSITION 2.1. *For all h and $p \neq 1$, $B_{p,h,n}$ is a solution of the MQYBE similar to Q_p.*

Proof. Define a shift operator $\phi_t\colon \mathbb{F}(z_1, z_2) \to \mathbb{F}(z_1, z_2)$ by $\phi_t \cdot f(z_1, z_2) = f(z_1 - t, z_2 - t)$ and let ϕ_t act as usual on operators by conjugation. Then, if $F_{p,t} = \phi_t \circ F_p$,

$$\phi_t \circ Q_p = F_{p,t} - \frac{(p^n - 1)(pz_2 + z_1 - t(p+1))}{(p^n + 1)(pz_2 - z_1 - t(p-1))}(F_{p,t} - P).$$

Choose $t = h/(p-1)$. Then $\phi_t \circ Q_p = B_{p,h,n}$. This shows that $B_{p,h,n}$ is similar to Q_p and, hence, satisfies the MQYBE when $p \neq 1$. □

Now the restriction of $B_{p,h,n}$ to $V_n \otimes V_n$ is a rational function of p which belongs to $\mathfrak{R}'$ and which for $p = 1$ is R_{p}. Thus R_{p} must be a "boundary solution" of the Yang–Baxter equation.

3. Vertex-IRF Transformations and Solutions of the Dynamical YBE

The original construction of the Cremmer–Gervais R-matrices was by a generalised kind of change of basis (a 'vertex-IRF transformation') from the Gervais–Neveu solution of the constant dynamical Yang–Baxter equation. Given the above construction of R_{p} as a rational degeneration of the Cremmer–Gervais matrices, it is natural to expect that R_{p} should be connected in the same way with some kind of rational degeneration of the Gervais–Neveu matrices. In fact this is precisely what happens. The appropriate solutions to the constant dynamical Yang–Baxter equation (DYBE) were found by Etingof and Varchenko in [7]. In classifying certain kinds of solutions to the constant DYBE, they found that all such solutions were equivalent to either a generalized form of the Gervais–Neveu matrix or to a rational version of this matrix. It turns out that R_{p} is connected via a vertex-IRF transformation with the simplest of this family of rational solutions to the constant DYBE.

Recall the framework for the dynamical Yang–Baxter equation given in [15]. Let H be a commutative cocommutative Hopf algebra. Let B be an H-module algebra with structure map $\sigma\colon H \otimes B \to B$. Denote by $\mathcal{C}$ the category of right H-comodules. Define a new category $\mathcal{C}_\sigma$ whose objects are right H-comodules but whose morphisms are $\hom_{\mathcal{C}_\sigma}(V, W) = \hom_H(V, W \otimes B)$ where B is given a trivial comodule structure. Composition of morphisms is given by the natural embedding of $\hom_H(V, W \otimes B)$ inside $\hom_H(V \otimes B, W \otimes B)$.

A tensor product $\tilde{\otimes}\colon \mathcal{C}_\sigma \times \mathcal{C}_\sigma \to \mathcal{C}_\sigma$ is defined on this category in the following way. For objects V and W, $V \tilde{\otimes} W$ is the usual tensor product of H comodules $V \otimes W$. In order to define the tensor product of two morphisms, define first for any H-comodule

W, a linear twist map $\tau\colon B \otimes W \to W \otimes B$ by

$$\tau(b \otimes w) = w_{(0)} \otimes \sigma(w_{(1)} \otimes b),$$

where $w \longmapsto \sum w_{(0)} \otimes w_{(1)}$ is the structure map of the comodule W. Then for any pair of morphisms $f\colon V \to V'$ and $g : W \to W'$, define

$$f \tilde{\otimes} g = (1 \otimes m_B)(1 \otimes \tau \otimes 1)(f \otimes g).$$

Etingof and Varchenko showed in [7, 8] that the bifunctor $\tilde{\otimes}$ makes $\mathcal{C}_\sigma$ into a tensor category. Let $V \in \mathcal{C}_\sigma$ For any $R \in \mathrm{End}_{\mathcal{C}_\sigma}(V \tilde{\otimes} V)$ we define elements of $\mathrm{End}_{\mathcal{C}_\sigma}(V \tilde{\otimes} V \tilde{\otimes} V)$, $R_{12} = R \tilde{\otimes} 1$ and $R_{23} = 1 \tilde{\otimes} R$. Then R is said to satisfy the σ-dynamical braid equation (σ-DBE) if $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$. If R is a solution of the σ-DBE then RP satisfies the σ-dynamical Yang–Baxter equation:

$$R_{12}R_{23}^{12}R_{12}^{123} = R_{23}R_{12}^{23}R_{23}^{132},$$

where, for instance, $R_{12}^{132} = P_{132}R_{12}P_{123}$.

A vertex-IRF transformation of a solution of the σ-DBE can then be defined [15, Section 3.3] as an invertible linear operator $A : V \to V \otimes B$ (that is, invertible in the sense of the composition of such operators defined above) such that the conjugate operator $R^A = A_2^{-1}A_1^{-1}RA_1A_2$ is a 'scalar' operator in the sense that $R^A(V \otimes V) \subset V \otimes V \otimes \mathbb{F}$. In this case R^A satisfies the traditional braid equation [15, Proposition 3.3]. Thus a vertex-IRF transformation transforms a solution of the σ-DYBE to a solution of the usual YBE.

Let T be the usual maximal torus of $\mathrm{SL}(n)$. Let V be the standard representation of $\mathrm{SL}(n)$ considered as a comodule over $H = \mathbb{F}[T]$ which we may consider as the group algebra of the weight lattice P; i.e., $H = \mathbb{F}[K_\lambda \mid \lambda \in P]$. Then V has a basis $\{e_i\}$ of weight vectors with weights v_i. Denote the structure map by $\rho : V \to V \otimes \mathbb{F}[T]$. Then $\rho(e_i) = e_i \otimes K_{v_i}$.

Let $S(\mathfrak{h}^*)$ be the symmetric algebra on $\mathfrak{h}^*$ and set $B = \mathrm{Frac}(S(\mathfrak{h}^*))$. Define an action $\sigma : H \otimes B \to B$ by

$$\sigma(K_\lambda \otimes v) = v - (\lambda, v).$$

Denote $\sigma(K_\lambda \otimes b)$ by b^λ. Recall that $(v_i, v_j) = \delta_{ij} - 1/n$. This fact will be used repeatedly in the calculations below.

Let R be the matrix $R_{\mathfrak{p}}$ defined in Section 1.1 with $h = 1/n$, considered as an operator on the space $V \otimes V$ where V has basis $\{e_1, \ldots, e_n\}$. Set $\tilde{R} = RP$ and let $\tilde{R}_{ij}^{kl}$ be the matrix coefficients of $\tilde{R}$ defined by $\tilde{R} \cdot e_i \otimes e_j = \sum_{k,l} \tilde{R}_{ij}^{kl} e_k \otimes e_l$. From Definition 1.3 we have that for any z_1 and z_2,

$$\sum_{k,l} \tilde{R}_{ij}^{kl} z_1^{k-1} z_2^{l-1} = \alpha(z_1 - z_2) z_1^{i-1} z_2^{j-1} + \beta(z_1 - z_2)(z_1 + 1/n)^{j-1}(z_2 - 1/n)^{i-1}.$$

where $\alpha(x) = 1/(x + 1/n)$ and $\beta(x) = 1 - \alpha(x)$. Define the operator $\mathcal{R} \in \mathrm{End}_{\mathcal{C}_\sigma} V \tilde{\otimes} V$

by

$$\begin{aligned}\mathcal{R}(e_i \otimes e_j) &= e_i \otimes e_j \otimes \alpha(v_i^{v_j} - v_j) + e_j \otimes e_i \otimes \beta(v_i^{v_j} - v_j)\\ &= e_i \otimes e_j \otimes \frac{1}{v_i - v_j + \delta_{ij}} + e_j \otimes e_i \otimes \left(1 - \frac{1}{v_i - v_j + \delta_{ij}}\right).\end{aligned}$$

This is the solution of the DBE corresponding to the standard example of solution of the DYBE of the type given in [7, Theorem 1.2]. Finally define an operator $A \in \operatorname{End}_{\mathcal{C}_\sigma}(V)$ by $A(e_i) = \sum e_k \otimes v_k^{i-1}$.

THEOREM 3.1. $\mathcal{R}^A = \tilde{R}$.

Proof. We prove that $\mathcal{R}A_1A_2 = A_1A_2\tilde{R}$. In matrix form this is equivalent to

$$\sum_{c,d} \mathcal{R}_{cd}^{ms} (A_i^c)^{v_d} A_j^d = \sum_{k,l} \tilde{R}_{ij}^{kl} (A_k^m)^{v_s} A_l^s.$$

Using the fact that $\beta(v_m^{v_s} - v_s) = 0$ when $m = s$

$$\begin{aligned}\sum_{k,l} \tilde{R}_{ij}^{kl} (A_k^m)^{v_s} A_l^s &= \sum_{k,l} \tilde{R}_{ij}^{kl} (v_m^{v_s})^{k-1} v_s^{l-1}\\ &= \alpha(v_m^{v_s} - v_s)(v_m^{v_s})^{i-1} v_s^{j-1} + \beta(v_m^{v_s} - v_s)\left(v_s - \frac{1}{n}\right)^{i-1} \left(v_m^{v_s} + \frac{1}{n}\right)^{j-1}\\ &= \alpha(v_m^{v_s} - v_s)(v_m^{v_s})^{i-1} v_s^{j-1} + \beta(v_m^{v_s} - v_s)(v_s^{v_m})^{i-1} (v_m)^{j-1}\\ &= \sum_{c,d} \mathcal{R}_{cd}^{ms} (A_i^c)^{v_d} A_j^d\end{aligned}$$

as required. □

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