

THE YANG-BAXTER EQUATION FOR OPERATORS ON FUNCTION FIELDS

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INTRODUCTION

In this note we show how infinite dimensional versions of the Cremmer-Gervais and Jordanian R -matrices occur as operators on rational function fields. Moreover we classify certain types of solutions of the Yang-Baxter equation on rational function fields and show that the above operators are essentially the only solutions.

Recall that if A is an integral domain and σ is an automorphism of A , then σ extends naturally to the field of rational functions $A(x)$ by acting on the coefficients. Denote by $\mathbb{F}(z_1, z_2)$ the field of rational functions in the variables z_1 and z_2 . Then for any $\sigma \in \text{Aut } \mathbb{F}(z_1, z_2)$, and any $i, j \in \{1, 2, 3\}$, we may define $\sigma_{ij} \in \text{Aut } \mathbb{F}(z_1, z_2, z_3)$ by realizing $\mathbb{F}(z_1, z_2, z_3)$ as $\mathbb{F}(z_i, z_j)(z_k)$. Set $\Gamma = \text{Aut } \mathbb{F}(z_1, z_2)$. Elements $R = \sum \alpha_i(z_1, z_2)\sigma_i$ of the group algebra $\mathbb{F}(z_1, z_2)[\Gamma]$ act as linear operators on $\mathbb{F}(z_1, z_2)$ and we may define in this way R_{ij} as linear operators on $\mathbb{F}(z_1, z_2, z_3)$. Thus we may look for solutions of the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ amongst such operators. Denote by P the operator $P \cdot f(z_1, z_2) = f(z_2, z_1)$.

First we look for operators of the form

$$R \cdot f(z_1, z_2) = \alpha(z_1/z_2)f(z_1, z_2) + \beta(z_1/z_2)f(z_2, z_1)$$

where $\alpha(x), \beta(x) \in \mathbb{F}(x)$. One solution of this is an infinite dimensional version of one member of the two parameter family of Cremmer-Gervais R -matrices [2, 12]. Moreover up to a certain natural equivalence, this is essentially the only such solution. This approach explains and clarifies the results on generating functions for the Cremmer-Gervais R -matrices in [11].

Next we consider the additive version; that is, operators of the form

$$R \cdot f(z_1, z_2) = \alpha(z_1 - z_2)f(z_1, z_2) + \beta(z_1 - z_2)f(z_2, z_1)$$

where $\alpha(x), \beta(x) \in \mathbb{F}(x)$. In this case the solution is again essentially unique. It is again a kind of direct limit of finite dimensional solutions of the Yang-Baxter equation. However this time the solutions are less familiar. They turn out (when twisted appropriately) to be generalizations of the Jordanian R -matrix [4] which may be viewed as quantizations of certain skew-symmetric solutions of the classical Yang-Baxter equation [5, 9, 10].

This work is an analog for the constant Yang-Baxter equation of work of Shibukawa and Ueno. In [14], Shibukawa and Ueno construct solutions of the Yang-Baxter

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equation with spectral parameter on meromorphic functions of two variables. They show that operators of the form

$$R(\lambda) = G(z_1 - z_2, \lambda)P - G(z_1 - z_2, \kappa)I$$

satisfy the equation

$$R_{12}(\lambda_1)R_{13}(\lambda_1 + \lambda_2)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1 + \lambda_2)R_{12}(\lambda_1)$$

for any $\kappa \in \mathbb{F}$ if G is of the form

$$G(z, \lambda) = \frac{\theta'(0)\theta(\lambda + z)}{\theta(\lambda)\theta(z)}$$

and θ satisfies the equation

$$\begin{aligned} &\theta(x + y)\theta(x - y)\theta(z + w)\theta(z - w) + \theta(x + z)\theta(x - z)\theta(y + w)\theta(w - y) \\ &+ \theta(x + w)\theta(x - w)\theta(y + z)\theta(y - z) = 0 \end{aligned}$$

The solutions of this equation are well known (see for instance, [15]). They are of three types: elliptic, trigonometric and rational. These lead to operators of the respective types which we shall denote $R_e(\lambda)$, $R_t(\lambda)$, and $R_r(\lambda)$. Felder and Pasquier showed that the elliptic solutions when twisted and restricted to certain finite dimensional spaces yield the Belavin R -matrices. A similar procedure when applied to $R_t(\lambda)$, and $R_r(\lambda)$ yields trigonometric and rational degenerations of the Belavin R -matrices. In these two degenerate cases it is possible to take the limit as $\lambda \rightarrow \infty$ to obtain solutions of the constant Yang-Baxter equation. For $R_t(\lambda)$ this yields (after the usual twisting and restriction) the Cremmer-Gervais R -matrices. For $R_r(\lambda)$ it yields the generalized Jordanian R -matrices referred to above. This situation can be summed up in the following way.

$$\begin{array}{ccc} R_e(\lambda) & & R_B(\lambda) \\ \downarrow & & \downarrow \\ R_t(\lambda) & \longrightarrow & R_t^\infty & \hat{R}_{CG}(\lambda) & \longrightarrow & R_{CG} \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ R_r(\lambda) & \longrightarrow & R_r^\infty & R_{B,r}(\lambda) & \longrightarrow & R_p \end{array}$$

The matrices on the left are the operators on meromorphic functions. The vertical arrows denote the degeneration of θ from a true theta function to a trigonometric function and then a rational function. The horizontal arrows denote passing to the limit as the spectral parameter tends to infinity. The right hand diagram gives the corresponding finite dimensional R -matrices.

1. THE CONSTANT YANG-BAXTER EQUATION ON RATIONAL FUNCTION FIELDS

We begin by looking at operators of the form

$$R \cdot f(z_1, z_2) = \alpha(z_1/z_2)f(z_2, z_1) + \beta(z_1/z_2)f(z_1, z_2)$$

where $\alpha(x), \beta(x) \in \mathbb{F}(x)$. We wish to determine for which functions α and β the operator R satisfies the Yang-Baxter equation. For a rational function $\alpha(x) \in \mathbb{F}(x)$, we shall denote by $\tilde{\alpha}$ the function $\alpha(z_1/z_2) \in \mathbb{F}(z_1, z_2)$. With this notation we may write R as $\tilde{\alpha}P + \tilde{\beta}I$.

Lemma 1.1. *The operator $R = \tilde{\alpha}P + \tilde{\beta}I$ satisfies the Yang-Baxter equation if and only if the following two equations are satisfied (in the rational function field $\mathbb{F}(x, y)$)*

$$(1.1) \quad \alpha(x)\alpha(y) = \alpha(xy^{-1})\alpha(y) + \alpha(x)\alpha(yx^{-1})$$

$$(1.2) \quad \alpha(x)\alpha(y)^2 + \beta(y)\beta(y^{-1})\alpha(xy) = \alpha(x)^2\alpha(y) + \beta(x)\beta(x^{-1})\alpha(xy)$$

Proof. Applying both sides of the Yang-Baxter equation to an arbitrary function $f(z_1, z_2, z_3)$ and comparing coefficients of $f(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$ for $\sigma \in \Sigma_3$ shows that the Yang-Baxter equation holds if and only if the following equations are satisfied.

$$\begin{aligned} \alpha(z_1/z_2)\alpha(z_2/z_3)^2 + \beta(z_2/z_3)\alpha(z_1/z_3)\beta(z_3/z_2) = \\ \alpha(z_1/z_2)^2\alpha(z_2/z_3) + \beta(z_1/z_2)\alpha(z_1/z_3)\beta(z_2/z_1) \end{aligned}$$

$$\alpha(z_2/z_3)\beta(z_1/z_2)\alpha(z_1/z_3) = \beta(z_1/z_2)\alpha(z_1/z_3)\alpha(z_2/z_1) + \alpha(z_1/z_2)\alpha(z_2/z_3)\beta(z_1/z_2)$$

$$\beta(z_2/z_3)\alpha(z_1/z_3)\alpha(z_3/z_2) + \beta(z_2/z_3)\alpha(z_2/z_3)\alpha(z_1/z_2) = \beta(z_2/z_3)\alpha(z_1/z_2)\alpha(z_1/z_3)$$

The last two equations are both equivalent to 1.1 and the first one to 1.2. \square

Notice that if $\alpha(x)$ and $\beta(x)$ are solutions of the above equations then so are $\lambda\alpha(x^n)$ and $\lambda\beta(x^n)$ for any integer n and any $\lambda \in \mathbb{F}$. Moreover we may clearly multiply $\beta(x)$ by any member of $\Gamma = \{g(x) \in \mathbb{F}(x) \mid g(x)g(x^{-1}) = 1\}$ and $\beta(x)$ will still satisfy 1.2. Surprisingly, the solutions to 1.1 and 1.2 turn out to be almost unique modulo these adjustments.

Theorem 1.2. *If $\alpha \neq 0$ the operator $R = \tilde{\alpha}P + \tilde{\beta}I$ satisfies the Yang-Baxter equation if and only if, there exist $\lambda \in \mathbb{F} \setminus 0$, $\mu \in \mathbb{F}$, $k \in \mathbb{Z}$ and $g(x) \in \Gamma$ such that*

$$\alpha(x) = \frac{\lambda}{1-x^k} \quad \text{and} \quad \beta(x) = \left(\frac{\lambda}{1-x^k} + \mu \right) g(x).$$

Proof. The verification that α and β satisfy the hypotheses of the lemma is elementary. Conversely, suppose that α satisfies 1.1. If a is a zero of α then equation 1.1 implies that $\alpha(ay^{-1})\alpha(y) = 0$ which is impossible unless $a = 0$. Thus we may write α in the form

$$\alpha(x) = \frac{x^m}{p(x)}$$

where $p(x)$ is a polynomial with $p(0) \neq 0$. If b is a root of p then equation 1.1 implies that

$$x^m p(by^{-1})[b^m p(yb^{-1}) - p(y)] = 0$$

whence $p(y) = b^m p(yb^{-1})$. Thus if a and b are roots of p so is ab^{-1} and the roots form a group H of order, say, k . Since $p(0) \neq 0$, we must have $b^m = 1$ for all $b \in H$. Hence $k \mid m$ and $p(x) = p(xb)$ for all $b \in H$. This implies that $p(x) = \lambda(1-x^k)^l$ for some integer l . Setting $y = x^{-1}$ in equation 1.1 and substituting yields $x^{m-lk} + x^m = (1+x^k)^l$ so $l = 1$ and $m = 0$ or k . This implies that α has the required form.

If α and β satisfy equations 1.1 and 1.2 then it is easily seen that

$$\beta(x)\beta(x^{-1}) = \alpha(x)\alpha(x^{-1}) + \gamma$$

for some scalar γ . Moreover for any fixed γ , any two solutions for β differ by a factor $g(x) \in \Gamma$. If α and β are as in the statement of the Theorem, then

$$\begin{aligned}\beta(x)\beta(x^{-1}) &= \alpha(x)\alpha(x^{-1}) + \mu(\alpha(x) + \alpha(x^{-1})) + \mu^2 \\ &= \alpha(x)\alpha(x^{-1}) + \mu\lambda + \mu^2.\end{aligned}$$

Thus for any α the β satisfying 1.2 are precisely the functions $\beta(x) = (\alpha(x) + \mu)g(x)$. \square

Remark 1. This result remains true in the case of meromorphic functions.

The most interesting example is the simplest one; when $k = 1$, $g(x) = 1$ and the constants are chosen so that RP satisfies the Hecke condition $(RP - q)(RP + q^{-1}) = 0$. That is,

$$\alpha(x) = \frac{\hat{q}}{1-x} \quad \text{and} \quad \beta(x) = \frac{q^{-1} - qx}{1-x}.$$

In this case R restricts to one of the Cremmer-Gervais R -matrices on certain finite dimensional subspaces.

Recall that the Cremmer-Gervais R -matrices are a two-parameter family of solutions to the constant Yang-Baxter equation. If V is an n -dimensional vector space with basis $\{e_1, e_2, \dots, e_n\}$, the Cremmer-Gervais operators are defined by [2, 12]

$$\rho_p(e_j \otimes e_i) = qp^{i-j}e_j \otimes e_i + \sum_k \hat{q}p^{i-k}\eta(i, j, k)e_k \otimes e_{i+j-k}$$

where q and p are non-zero elements of the base field \mathbb{F} , $\hat{q} = q - q^{-1}$ and

$$\eta(i, j, k) = \begin{cases} 1 & \text{if } i \leq k < j \\ -1 & \text{if } j \leq k < i \\ 0 & \text{otherwise} \end{cases}$$

Proposition 1.3. *Let R be the operator $\tilde{\alpha}P + \tilde{\beta}I$ where $\alpha(x) = \hat{q}/(1-x)$ and $\beta(x) = (q^{-1} - qx)/(1-x)$. Let V be an n -dimensional vector space and identify $V \otimes V$ with the subspace of $\mathbb{F}(z_1, z_2)$ of polynomials of degree less than or equal to n in both variables. Then R leaves $V \otimes V$ invariant and the matrix representing R with respect to the basis $z_1^j z_2^i$ is ρ_1 .*

Proof. Notice that for any i and j ,

$$\sum_k \eta(i, j, k)x^k = \frac{x^i - x^j}{1-x}$$

From this it follows easily that

$$R \cdot z_1^j z_2^i = qz_1^j z_2^i + \sum_k \eta(i, j, k)z_1^k z_2^{i+j-k}$$

as required. \square

Just as the more general Cremmer-Gervais operators ρ_p may be obtained from ρ_1 by a simple twisting method, so we may twist this operator R to obtain more general matrices R_p which restrict to ρ_p in the same way as R restricts to ρ_1 . Fix $p \in \mathbb{F} \setminus 0$, and define an invertible operator F on $\mathbb{F}(z_1, z_2)$ by

$$F \cdot f(z_1, z_2) = f(p^{-1}z_1, pz_2)$$

Theorem 1.4. *Let $R = \tilde{\alpha}P + \tilde{\beta}I$ be a solution of the Yang-Baxter equation on $\mathbb{F}(z_1, z_2)$. Then $R_F = F_{21}^{-1}RF_{12}$ is also a solution of the Yang-Baxter equation. Moreover, when $\alpha(x) = \hat{q}/(1-x)$ and $\beta(x) = (q^{-1} - qx)/(1-x)$, then R_F again leaves $V \otimes V$ invariant and the matrix representing R_F with respect to the basis $z_1^j z_2^i$ is ρ_p .*

Proof. For the first assertion it suffices to check the relations

1. $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$
2. $R_{12}F_{23}F_{13} = F_{13}F_{23}R_{12}$
3. $R_{23}F_{12}F_{13} = F_{13}F_{12}R_{23}$

The second assertion is verified exactly as in the previous proposition. \square

We now turn briefly to consider the ‘‘additive’’ version of this construction. Namely we can look for operators of the form

$$R \cdot f(z_1, z_2) = \alpha(z_1 - z_2)f(z_2, z_1) + \beta(z_1 - z_2)f(z_1, z_2)$$

satisfying the Yang-Baxter equation. One sees analogously that R satisfies the Yang-Baxter equation if and only if

$$\alpha(x)\alpha(y) = \alpha(x-y)\alpha(y) + \alpha(x)\alpha(y-x)$$

$$\alpha(x)\alpha(y)^2 + \beta(y)\beta(-y)\alpha(x+y) = \alpha(x)^2\alpha(y) + \beta(x)\beta(-x)\alpha(x+y)$$

Theorem 1.5. *The operator $R = \check{\alpha}P + \check{\beta}I$ satisfies the Yang-Baxter equation if and only if there exist $\lambda, \mu \in \mathbb{F}$, and $g(x) \in \Gamma'$ such that*

$$\alpha(x) = -\frac{\lambda}{x} \quad \text{and} \quad \beta(x) = \left(\mu + \frac{\lambda}{x}\right)g(x).$$

For this operator,

$$R \cdot z_1^j z_2^i = \mu z_1^j z_2^i + \sum_k \eta(i, j, k) z_1^k z_2^{i+j-k-1}$$

Hence R again leaves the subspaces $V \otimes V$ invariant and on these spaces becomes the operator

$$R(e_j \otimes e_i) = \mu e_j \otimes e_i + \sum_k \lambda \eta(i, j, k) e_k \otimes e_{i+j-k-1}.$$

One can again twist using an additive version of the twist described in 1.4. Restricting these matrices to finite dimensional subspaces yields quantizations of the boundary classical r -matrices discovered by Gerstenhaber and Giaquinto [9, 10] (see [5] for details).

If we replace $\mathbb{F}(z_1, z_2)$ by meromorphic functions in two variables, then replacing x by $e^{2\pi iz}$ in the functions of Theorem 1.2 clearly leads to solutions of the additive form. In particular the simplest such solution yields

$$\alpha(z) = \frac{\sin \pi h}{\sin \pi z}, \quad \beta(z) = \frac{\sin \pi(h+z)}{\sin \pi z}$$

Because R is Hecke, The family of operators

$$R(\lambda) = e^{-\pi i \lambda} R - e^{\pi i \lambda} R^{-1}$$

will satisfy the general Yang-Baxter equation. After multiplying by a suitable scalar, R will be of the particularly symmetric form $R(\lambda) = \check{\alpha}I + \check{\beta}P$ where

$$\alpha(z) = \frac{\sin \pi(z + \lambda)}{\sin \pi z} \sin \pi \lambda, \quad \beta(z) = \frac{\sin \pi(z + h)}{\sin \pi z} \sin \pi h.$$

This is the trigonometric version of the R operator of Shibukawa and Ueno discussed in the introduction.

Finally let us note an interesting connection between the symmetric algebra associated to infinite dimensional Cremmer-Gervais operator and the affine quantum universal enveloping algebra $U_q(\mathfrak{sl}(\hat{2}))$. Let $V = \mathbb{F}[z]$ and let R be the operator on $V \otimes V$ described in 1.3. As usual we can form the associated q -symmetric algebra as $S_{R,q}(V) = T(V)/((RP - q)(V \otimes V))$. Now for any $f \in V \otimes V = \mathbb{F}[z_1, z_2]$, we have $(R - q)f = \beta(P - I)f$. From this it follows that the elements of $(R - q)(V \otimes V)$ are of the form $(qz_2 - q^{-1}z_1)g$ where $g(z_1, z_2)$ is symmetric (in the usual sense). Now,

$$(qz_2 - q^{-1}z_1)(z_1^k z_2^l + z_1^l z_2^k) = qz_1^k z_2^{l+1} - q^{-1}z_1^{k+1} z_2^l + qz_1^l z_2^{k+1} - q^{-1}z_1^{l+1} z_2^k.$$

Hence if X_k denotes the image of z^k in $S_{R,q}(\mathbb{F}[z])$, then the relations defining the symmetric algebra are

$$X_{k+1}X_l - q^2X_kX_{l+1} = q^2X_lX_{k+1} - X_{l+1}X_k$$

These are precisely the relations defining the subalgebra U^+ of $U_q(\mathfrak{sl}(\hat{2}))$ [3]. Hence we have $S_{R,q}(\mathbb{F}[z]) \cong U^+$. It would be interesting to have a more abstract explanation for this isomorphism.

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