ON THE CREMMER-GERVAIS QUANTIZATIONS OF $SL(n)$

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Abstract. The non-standard quantum groups $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ and $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$ are constructed for a two parameter version of the Cremmer-Gervais R-matrix. An epimorphism is constructed from $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ onto the restricted dual $U_{\bar{R}}(\mathfrak{gl}(\mathfrak{n}-1))$ associated to a related smaller R-matrix of the same form.

A related result is proved concerning factorizable Lie bialgebras. For any such Lie bialgebra, the dual Lie bialgebra has a canonical homomorphic image which is again factorizable.

1. Introduction

We define and study some non-standard quantifications $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$ and $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ of the algebra of functions on the special and general linear groups. These algebras are constructed using a two parameter version of the nonstandard solutions of the Yang-Baxter equation discovered by Cremmer and Gervais [7]. We refer to these algebras as Cremmer-Gervais quantum groups. Our main result states that there is a homomorphism from $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ onto the dual, or quantized universal enveloping algebra, $U_{\bar{R}}(\mathfrak{gl}(\mathfrak{n}-1))$ associated to a related smaller R-matrix of the same form. This result implies that the algebraic structure of $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$ will be significantly different from that of the standard quantification $\mathbb{C}_{\shortparallel}[\mathbb{SL}(\ltimes)]$. For instance, in the case when $n = 3$, the main result provides a map from $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\nvDash)]$ onto one of the multi-parameter versions of $U_q(\mathfrak{gl}(2))$. Thus while all the simple finite dimensional modules over $\mathbb{C}_{\shortparallel}[\mathbb{SL}(\nvDash)]$ are one dimensional, $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\nvDash)]$ has simple modules of all possible dimensions.

In contrast to the enormous literature on the standard quantum groups, very little attention has been paid to these non-standard examples. In their original paper, Cremmer and Gervais define and discuss the bialgebra $A(R)$ associated to their R-matrices. They also define and discuss the associated symmetric space $S(R)$. In [1, 2] Balog, Dabrowski and Fehér define the quantum determinant and antipode needed for the construction of $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\nvDash)]$. They also define and study the associated quantized universal enveloping algebra $U_R(\mathfrak{sl}(3))$. The associated formal deformation was studied by Gerstenhaber, Giaquinto and Schack in [11].

This paper represents another step in the project to understand the algebraic structure of quantifications of $\mathbb{C}[\mathbb{G}]$ for G a connected semi-simple group. In the standard and multi-parameter case the classification of the primitive ideals was achieved in the series of papers [15, 16, 17, 18, 19]. Our main result implies that the primitive spectrum of $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ contains the primitive spectrum of $U_{\bar{R}}(\mathfrak{gl}(\mathfrak{n}-1))$ as a closed subset and hence that any classification of primitive ideals of $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ should include a classification of the primitive ideals of $U_R(\mathfrak{gl}(\mathfrak{n}))$.

In the final section we give a result on the dual of a factorizable Lie bialgebra that to some extent 'explains' the main results of section 4 and suggests how these results might be broadened to more general quantifications of connected semi-simple groups

associated to solutions of the modified classical Yang-Baxter equation (MCYBE). Let $\mathfrak{g}$ be a semi-simple complex Lie algebra with a coboundary Lie bialgebra structure given by a solution r of the MCYBE and let $\mathfrak{g}^*$ be the dual Lie bialgebra. Then there is a canonical homomorphic image $\tilde{\mathfrak{g}}$ of $\mathfrak{g}^*$ which is reductive and for which the Lie bialgebra structure is given again by a solution $\tilde{r}$ of the MCYBE. In particular when r is the 'Cremmer-Gervais' classical r-matrix for $\mathfrak{sl}(\mathfrak{n})$, then $\tilde{r}$ is one of the two parameter family of Cremmer-Gervais r-matrices for $\mathfrak{gl}(\mathfrak{n}-\mathfrak{1})$. Thus our main result on Cremmer-Gervais quantum groups may be viewed as a quantification of this special case of Theorem 6.4.

Many of the ideas in this paper were developed in discussions with T. Levasseur. We thank him for his very significant contributions.

2. Braided Hopf algebras and their duals

We first view a general context for our main result. We show that the restricted dual A° of a braided Hopf algebra A contains a canonical braided subalgebra U_0. Hence there is a natural map from A into the dual of U_0. In this and the next section we consider Hopf algebras over an arbitrary field, k.

Recall the definition of a braided bialgebra given in [21]. (Notice that this definition is slightly different from that given in [8, 20] and that in the latter such algebras are termed *cobraided*).

Definition . A braiding on a bialgebra A is a bilinear pairing $\langle\ \mid\ \rangle$ such that the following conditions hold for all a, b, and c in A

1. $\sum\langle a_{(1)} \mid b_{(1)}\rangle b_{(2)}a_{(2)} = \sum\langle a_{(2)} \mid b_{(2)}\rangle a_{(1)}b_{(1)}$
2. $\langle\ ,\ \rangle$ is invertible in $(A\otimes A)^*$;
3. $\langle a \mid bc\rangle = \sum\langle a_{(1)} \mid b\rangle\langle a_{(2)} \mid c\rangle$.
4. $\langle ab \mid c\rangle = \sum\langle b \mid c_{(1)}\rangle\langle a \mid c_{(2)}\rangle$

Recall that in this situation, we also have that $\langle a \mid 1\rangle = \epsilon(a) = \langle 1 \mid a\rangle$ for all $a \in A$.

A braided Hopf algebra is a Hopf algebra which is braided as a bialgebra. The antipode of a braided Hopf algebra is always bijective by [8, Theorem 1.3]. Moreover, in this case the braiding is S-invariant in the sense that $\langle a \mid b\rangle = \langle S(a) \mid S(b)\rangle$.

For any braided Hopf algebra A, the braiding induces a pair of Hopf algebra maps $l^\pm : A^{op} \to A^\circ$ given by

$$l^+(a)(b) = \langle a \mid b\rangle \quad \text{and} \quad l^-(a)(b) = \langle b \mid S(a)\rangle$$

Denote by $U^\pm$ the images of $l^\pm$ respectively. Then $U^\pm$ are obviously Hopf subalgebras of A°. Define the FRT dual $U(A)$ to be the Hopf subalgebra generated by U^+ and U^-. The braiding on A induces a Hopf pairing on $U^+\otimes(U^-)^{op}$ given by $\pi(l^+(a), l^-(b)) = \langle a \mid b\rangle$ using which one can construct the Drinfeld double $U^+\bowtie U^-$. The multiplication map $u\otimes v \mapsto uv$ is then a Hopf algebra map from $U^+\bowtie U^-$ to $U(A)$. Set

$$U_0(A) = U^+\cap U^-$$

Then $U_0(A)$ is also a Hopf subalgebra of A°. When there is no danger of ambiguity we shall denote $U_0(A)$ by U_0.

Proposition 2.1. *Consider the pairing on U_0 induced from the pairing π^{-1} on $(U^+)^{op}\otimes U^-$. That is, if $x = l^+(a)$ and $y = l^-(b)$ are elements of U^0, then*

$$(x \mid y) = \langle a \mid S(b)\rangle$$

This pairing is a braiding on U_0.

Proof. The last three conditions follow easily from the definition of the pairing. Condition (1) follows from the definition of multiplication in the Drinfeld double and the fact that the multiplication map is a homomorphism. □

Thus the FRT dual $U(A)$ of a braided Hopf algebra A contains a canonical braided Hopf subalgebra $U_0(A)$. Conversely the braided Hopf algebra has a canonical homomorphic image which contains the FRT dual of $U_0(A)$:

Theorem 2.2. *Let A be a braided Hopf algebra and let B be a braided Hopf subalgebra of $U_0(A)$. There is a natural Hopf algebra map $\phi : A \to B^\circ$. If $a \in A$ is such that $l^\pm(a) \in B$, then*

$$\phi(a) = l_B^\pm \circ l_A^\pm(a).$$

Hence $\phi(A) \supset U(B)$.

Proof. The map ϕ is given by $\phi(a)(u) = u(a)$. Suppose that $a \in A$ is such that $l^+(a) \in U_0$ and let $u = l^-(b) \in U_0$. Then

$$\phi(a)(u) = l^-(b)(a) = \langle a \mid S(b)\rangle = (l^+(a) \mid l^-(b)) = l_B^+(l_A^+(a))(u)$$

The result for $l^-(a)$ is proved similarly. □

Example . In the case of standard quantum groups, these results are not particularly interesting. Let G be a connected, simply connected semi-simple complex algebraic group and let $\mathfrak{g} = \operatorname{Lie}\mathfrak{G}$. Following the notation of [17], let $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ be the usual quantum group and quantized universal enveloping algebra. Then the algebra U_0 is the usual 'Cartan part' U^0 with braiding given by restriction of the Rosso form. The map $\phi : \mathbb{C}_q[G] \to (U^0)^\circ$ becomes an identification of the undeformed algebra of functions on the maximal torus $\mathbb{C}[\mathbb{H}]$ with the FRT dual of U^0.

We shall also need the following presumably well-known observation.

Proposition 2.3. *Suppose that A and B are braided Hopf algebras. Let $\chi : A \to B$ be a surjective Hopf algebra map which is braided in the sense that $\langle a, a'\rangle = \langle \chi(a) \mid \chi(a')\rangle$. The induced map $\chi^* : U(B) \to U(A)$ is an isomorphism.*

3. Multi-parameter R-matrices and twisted braided bialgebras

In this section we outline some results concerning the construction of a family of multi-parameter R-matrices from a given R-matrix. None of the ideas here are particularly new. Similar ideas are used in [6, 17, 22, 23, 24]. The twists described below are special cases of the dual of Drinfeld's gauge transformations [10, 23]. The cocycle condition ensures that the associator is still trivial.

Definition . Let A be a Hopf algebra. A 2-cocycle on A is an invertible pairing $\sigma : A \otimes A \to k$ such that for all x, y and z in A,

$$\sum \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sum \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

and $\sigma(1, 1) = 1$.

Given a 2-cocycle σ on a Hopf algebra, one can twist the multiplication to get a new Hopf algebra A_σ. The new multiplication is given by

$$x \cdot y = \sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \sigma^{-1}(x_{(3)}, y_{(3)}).$$

See [9] for further details.

Theorem 3.1. *Let A be a braided bialgebra with braiding β. Let σ be a 2-cocycle on A. Let A_σ be the twisted bialgebra defined above. Then $\sigma\beta(\sigma^\top)^{-1}$ (convolution product) is a braiding on A_σ.*

Proof. One can verify directly that $\sigma\beta(\sigma^\top)^{-1}$ satisfies the axioms of a braiding. Similar results are proved in [6, 22, 23, 24]. □

Suppose that an invertible operator $R : V \otimes V \to V \otimes V$ is a solution of the Yang Baxter equation (YBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Let P be the twist operator $P(v \otimes v') = v' \otimes v$. Then the operator RP is a Yang Baxter operator in the sense of [21]; i.e., an invertible solution of

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

We will denote by $A(R)$ the braided bialgebra denoted by $A(RP)$ in [21]. Denote by T_i^k the standard generators. The relations in $A(R)$ are then

$$\sum_{u,v} R_{ji}^{uv} T_u^k T_v^l = \sum_{u,v} T_i^u T_j^v R_{vu}^{kl}$$

and the braiding is given by

$$\langle T_i^k \mid T_j^l \rangle = R_{ji}^{lk}$$

For a given basis $\{e_i \mid i = 1, \ldots, n\}$ for V, define $e_{ij} : V \to V$ by $e_{ij}(e_k) = \delta_{ik} e_j$. Then we may write $R = \sum R_{ij}^{kl} e_{ik} \otimes e_{jl}$.

Suppose that $\sigma \in (A \otimes A)^*$. Then for any left comodule V over A there is an induced endomorphism $\sigma_V \in \operatorname{End} V \otimes V$ given by

$$\sigma_V(v \otimes v') = \sigma(v_{(1)}, v'_{(1)}) v_{(2)} \otimes v'_{(2)} \tag{3.1}$$

(see, for example [21]).

Corollary 3.2. *Let $A = A(R)$ where R is an invertible solution of the Yang Baxter equation. Let σ be a 2-cocycle on $A(R)$. Set $Q = \sigma_V$ and let $R_\sigma = QR(PQ^{-1}P)$. Then R_σ is a solution of the YBE and $A(R)_\sigma \cong A(R_\sigma)$ as braided bialgebras.*

Proof. This follows from the universal properties of $A(R)$ given in [21]. □

Note that for the corresponding Yang Baxter operators $\check{R}_\sigma = R_\sigma P$ and $\check{R} = RP$ we have the simpler relation $\check{R}_\sigma = Q\check{R}Q^{-1}$.

Theorem 3.3. *Suppose that R is a solution of the YBE such that*

$$R_{ij}^{kl} \neq 0 \Rightarrow i + j = k + l.$$

Let $p \in \mathbb{C}$ and let $Q = \sum p^{j-i} e_{ii} \otimes e_{jj}$. Then QRQ is also a solution of the YBE.

Proof. There is a bialgebra map from $A(R)$ into the functions on the 2-torus $B = k[\alpha^{\pm 1}, \beta^{\pm 1}]$ given by $T_i^j \mapsto \alpha\beta^i\delta_{ij}$. There is a 2-cocycle on B given by $\sigma(\alpha^i\beta^j, \alpha^k\beta^l) = p^{i+l-j-k}$ [17]. This induces a 2-cocycle on $A(R)$ such that $\sigma(T_i^k, T_j^l) = p^{j-i}\delta_{ik}\delta_{jl}$. Since $PQ^{-1}P = Q$, the result follows from the previous corollary. □

Remark . Cotta-Ramusino and Rinaldi have a similar result (Thm 2.1). No proof is given and this result appears only to hold for the standard R-matrices.

4. Multi-parameter Cremmer-Gervais R-matrices

We now apply the results of the previous section to construct a two-parameter family of solutions of the YBE from the one-parameter family given in [7]. For this and the following section, we shall assume that the base field is the complex numbers. Let $0 \neq q, p \in \mathbb{C}$. Set

$$\begin{aligned} R = R_n(q,p) = p^{-1}\Bigg(&\sum_{i=1}^{n} q e_{ii}\otimes e_{ii} + q\sum_{i>j} p^{-2(i-j)} e_{ii}\otimes e_{jj} \\ &+ q^{-1}\sum_{i<j} p^{-2(i-j)} e_{ii}\otimes e_{jj} \\ &+ (q^{-1}-q)\sum_{i<j}\sum_{k=1}^{j-i-1} p^{2k} e_{i,j-k}\otimes e_{j,i+k} \\ &+ (q-q^{-1})\sum_{i>j}\sum_{k=0}^{i-j-1} p^{-2k} e_{i,j+k}\otimes e_{j,i-k}\Bigg) \end{aligned} \tag{4.1}$$

When $q = p^n$, this is the R-matrix given in [7]. The following description of the matrix R_{ij}^{kl} is useful for some of the calculations needed later.

$$pR_{ij}^{kl} = \begin{cases} qp^{2(l-i)}, & \text{if } i=k\geq j=l \\ q^{-1}p^{2(l-i)}, & \text{if } i=k<j=l \\ (q-q^{-1})p^{2(l-i)} & \text{if } j\leq k<i,\ i+j=k+l \\ (q^{-1}-q)p^{2(l-i)} & \text{if } i<k<j,\ i+j=k+l \end{cases} \tag{4.2}$$

Notice that R satisfies the homogeneity condition:

$$R_{ij}^{kl} \neq 0 \Rightarrow i+j=k+l.$$

Hence we may apply Theorem 3.3.

Theorem 4.1. *The operator $R = R_n(q,p)$ satisfies the Yang Baxter equation for all $q, p \in \mathbb{C}^\times$. The Yang Baxter operator $\check{R} = pRP$ satisfies the Hecke equation:*

$$(\check{R}-q)(\check{R}+q^{-1}) = 0.$$

Proof. Set $Q(p) = \sum p^{j-i} e_{ii}\otimes e_{jj}$. Then for any q, p and p',

$$p'R(q,pp') = Q(p')R(q,p)Q(p')$$

The fact that $R(q,p)$ satisfies the YBE when $q = p^n$ is proved in [7]. It then follows from 3.3 that $R(q,p)$ satisfies the YBE for all values of p. The fact that $\check{R}$ satisfies the Hecke equation can be calculated directly or deduced from the one parameter case using the above identity. □

The corresponding symmetric and exterior algebras are the graded algebras defined by

$$S(R) = \mathbb{C}\langle \curvearrowleft_{\ltimes}, \ldots, \curvearrowleft_{\ltimes}\rangle / (\{ \curvearrowleft_{\ltimes}\curvearrowleft_{\ltimes}\curvearrowleft_{\ltimes} - \sum_{\urcorner,\lessdot} \mathbb{R}^{\urcorner\lessdot}_{\urcorner\lessdot} \curvearrowleft_{\urcorner}\curvearrowleft_{\lessdot}\})$$

and

$$\Lambda(R) = \mathbb{C}\langle \curvearrowleft_{\aleph}, \ldots, \curvearrowleft_{\ltimes} \rangle / (\{ {}_{\shortparallel}{}^{-\aleph} \curvearrowleft_{\beth} \curvearrowleft_{\beth} + \sum_{\daleth, \lessdot} {}_{\shortparallel}\mathbb{R}^{\lessdot}_{\beth\beth} \curvearrowleft_{\daleth} \curvearrowleft_{\lessdot} \})$$

respectively. For a graded algebra B, denote by $P_B(t)$ the Poincare series of B.

Theorem 4.2. *The algebra $\Lambda(R)$ is generated by elements $x_1, \ldots, x_n$ subject to the relations*

$$x_i^2 = 0, \quad x_i x_j = -p^{2(j-i)} x_j x_i$$

The Poincare series of the above algebras are the same as the commutative case:

$$P_S(t) = (1-t)^{-n}, \quad P_\Lambda(t) = (1+t)^n, \quad P_A(t) = (1-t)^{-n^2}$$

Proof. In the one parameter case this description of $\Lambda(R)$ is given in [14]. The proof in the multiparameter case is similar. The formula for $P_\Lambda(t)$ is then clear. The formula for $P_S(t)$ then follows from [12]. The formula for $P_A(t)$ can be deduced from the one-parameter case [14] using 3.2. □

This result implies that $A(R)$ has a group-like q-determinant element which can be used to construct the quantum group $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ and in the one-parameter case, $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$. Since $\Lambda^n(R)$ is a one dimensional $A(R)$-comodule generated by $x_1 \ldots x_n$, there exists a group like element $\det_q$ such that

$$\rho(x_1 \ldots x_n) = \det_q \otimes x_1 \ldots x_n.$$

where $\rho : \Lambda^n(R) \to A \otimes \Lambda^n(R)$ is the comodule structure map.

Proposition 4.3.
1. *$det_q = \sum_{\sigma \in S_n} a_\sigma T_1^{\sigma(1)} \ldots T_n^{\sigma(n)}$ where a_σ is the scalar such that in $\Lambda(R)$, $x_{\sigma(1)} \ldots x_{\sigma(n)} = a_\sigma x_1 \ldots x_n$.*
2. *For all i and k,*
$$\langle det_q \mid T_i^k \rangle = (q^{-1}p^n)^{(n-2i)} \delta_{ik} \quad \text{and} \quad \langle T_i^k \mid det_q \rangle = (q^{-1}p^n)^{-(n-2i+2)} \delta_{ik}$$
3. *For all i and k,*
$$T_i^k det_q = (q^{-1}p^n)^{2(i-k)} det_q T_i^k$$
Hence det_q is a normal element of $A(R)$.

Proof. For the proof of part (1) observe that

$$\begin{aligned} \rho(x_1 \ldots x_n) &= \left(\sum T_1^j \otimes x_j \right) \ldots \left(\sum T_n^j \otimes x_j \right) \\ &= \left(\sum_{\sigma \in S_n} T_1^{\sigma(1)} \ldots T_n^{\sigma(n)} \otimes x_{\sigma(1)} \ldots x_{\sigma(n)} \right) \\ &= \left(\sum_{\sigma \in S_n} a_\sigma T_1^{\sigma(1)} \ldots T_n^{\sigma(n)} \right) \otimes x_1 \ldots x_n. \end{aligned}$$

To prove the second assertion in part (2), observe that

$$\begin{aligned} \langle T_i^k \mid \det_q \rangle &= \sum_{\sigma \in S_n} a_\sigma \sum_{j_1, \ldots, j_{n-1}} \langle T_i^{j_1} \mid T_1^{\sigma(1)} \rangle \ldots \langle T_{j_{n-1}}^k \mid T_n^{\sigma(n)} \rangle \\ &= \sum_{\sigma \in S_n} a_\sigma \sum_{j_1, \ldots, j_{n-1}} R_{i,1}^{j_1 \sigma(1)} \ldots R_{j_{n-1}, n}^{k, \sigma(n)} \end{aligned}$$

Now the homogeneity of R immediately implies that the right hand side is zero unless $i = k$. Assume therefore that $i = k$. Suppose that there exists a $\sigma \neq e$ and a multi-index $\{j_1, \dots j_{n-1}\}$ such that the term

$$R_{i1}^{j_1\sigma(1)} \dots R_{j_{n-1},n}^{k,\sigma(n)}$$

is non-zero. Let u be the smallest integer such that $\sigma(u) > u$. Then for all $v < u$, we have $\sigma(v) = v$ and $j_v = i$. Since

$$\langle T_{j_{u+1}}^{j_u} \mid T_u^{\sigma(u)} \rangle \neq 0$$

we must have $j_u < i = j_{u-1}$, so $u \leq j_u < i$. It then follows by induction that $j_v \geq u$ for $v \geq u$. Now pick $v > u$ such that $\sigma(v) = u$. Then

$$0 \neq \langle T_{j_{v-1}}^{j_v} \mid T_v^u \rangle \Rightarrow j_{v-1} < u < v$$

contradicting the previous assertion. Thus

$$\langle T_i^k \mid \det_q \rangle = \prod_{j=1}^{n} R_{ji}^{ji} \delta_{ik} = (q^{-1}p^n)^{-(n-2i+2)} \delta_{ik}$$

as required. The proof of the first assertion is analogous.

Since $\det_q$ is group-like, the first braiding axiom implies that

$$\sum \langle \det_q \mid T_i^m \rangle T_m^k \det_q = \sum \langle \det_q \mid T_m^k \rangle \det_q T_i^m$$

The third assertion then follows from part (2). □

Definition . By the proposition $\det_q$ is a normal element and hence we may localize with respect to the Ore set $\{\det_q^k \mid k \in \mathbb{N}\}$. Denote the algebra $A(R)[\det_q^{-1}]$ by $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$. In the case where $p^n = q$, $\det_q$ is central. Denote the algebra $A(R)/(\det_q - 1)$ by $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$.

Theorem 4.4. *The algebra $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ has a braided Hopf algebra structure such that the natural map $A(R) \to \mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ is a morphism of braided bialgebras.*

Suppose that $p^n = q$. Then $(det_q - 1)$ is a braided Hopf ideal of $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ and $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$ is a braided Hopf algebra.

Proof. By [13, 3.1], the localization $A(R)[\det_q^{-1}]$ has a natural braided bialgebra structure. On the other hand Gurevitch [12, 5.10] proves that this localization has an antipode and is therefore a Hopf algebra. The second assertion is clear. □

5. The main theorem

We now investigate the maps defined in the first section in the case where $A = \mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ and R is the 2-parameter Cremmer-Gervais R-matrix. In this case we denote $U(\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)])$ by $U_R(\mathfrak{gl}(\mathfrak{n}))$. (Note that since this is the FRT dual, it is not the standard definition of a quantized universal enveloping algebra of $\mathfrak{gl}(\mathfrak{n})$). Recall from [21] that $\langle T_i^k \mid T_j^l \rangle = R_{ji}^{lk}$ and $\langle T_i^k \mid S(T_j^l) \rangle = (R^{-1})_{ji}^{lk}$. Hence

$$l^+(T_i^k)(T_j^l) = R_{ji}^{lk}, \qquad l^+(T_i^k)(S(T_j^l)) = (R^{-1})_{ji}^{lk}$$

and

$$l^-(T_i^k)(T_j^l) = (R^{-1})_{ij}^{kl}, \qquad l^-(T_i^k)(S(T_j^l)) = R_{ij}^{kl}$$

Theorem 5.1. *The two parameter R-matrix $R(q,p)$ satisfies:*

1. $R(q,p)^{-1} = R(q^{-1}, p^{-1})$

2. $(R(q,p)^{-1})^{kl}_{ij} = R(q,p)^{l,k+1}_{j,i+1}$ *for* $i,k<n$ *and all* j *and* l.
3. $R_n(q,p)^{k+1,l+1}_{i+1,j+1} = R_{n-1}(q,p)^{kl}_{ij}$

Proof. The Hecke relation

$$\check{R}^2 = (q-q^{-1})\check{R} + I$$

implies that

$$R^{-1} = p^2 PRP - p(q-q^{-1})P$$

From this the first assertion can be proved by direct calculation. The remaining assertions are verified directly using (4.2). □

The following result is given in the case $n=3$ in [1]

Theorem 5.2. 1. *For all* $i>1$, $l^+(T^1_i)=0$.
2. *For all* $i<n$, $l^-(T^n_i)=0$.
3. *For all* $i,k=1,\dots,n-1$, $l^-(T^k_i) = l^+(T^{k+1}_{i+1})$.

Thus the elements $t^k_i = l^-(T^k_i) = l^+(T^{k+1}_{i+1})$ *belong to* U_0. *Moreover,*

$$(t^k_i \mid t^l_j) = R^{kl}_{ij}$$

for all $i,k=1,\dots,n-1$.

Proof. Recall that $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$ is generated by the T^k_i and the $S(T^k_i)$). For any j and l,

$$l^+(T^1_i)(T^l_j) = R^{l1}_{ji} = 0 \text{ if } i \neq 1$$

and

$$l^+(T^1_i)(S(T^l_j)) = (R^{-1})^{l1}_{ji} = 0 \text{ if } i \neq 1$$

An induction argument then shows that for $i>1$, the $l^+(T^1_i)$ vanish on $\mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]$, proving (1). A similar argument works for part (2) using

$$l^-(T^n_i)(T^l_j) = (R^{-1})^{nl}_{ij} = 0 \text{ if } i \neq n.$$

and

$$l^-(T^n_i)(S(T^l_j)) = (R)^{nl}_{ij} = 0 \text{ if } i \neq n.$$

To show the equality in part (3), it suffices similarly to show that the action of the elements $l^-(T^k_i)$ and $l^+(T^{k+1}_{i+1})$ coincide on the generators. Using Theorem 5.1, we see that

$$l^+(T^{k+1}_{i+1})(T^l_j) = R^{l,k+1}_{j,i+1} = (R^{-1})^{kl}_{ij} = l^-(T^k_i)(T^l_j)$$

and

$$l^+(T^{k+1}_{i+1})(S(T^l_j)) = (R^{-1})^{l,k+1}_{j,i+1} = R^{kl}_{ij} = l^-(T^k_i)(S(T^l_j))$$

as required. Finally,

$$(t^k_i \mid t^l_j) = (l^+(T^{k+1}_{i+1}) \mid l^-(T^l_j)) = \langle T^{k+1}_{i+1} \mid S(T^l_j)\rangle = (R^{-1})^{l,k+1}_{j,i+1} = R^{kl}_{ij}.$$

□

Recall that if R is a solution of the Yang-Baxter equation, then so is $R' = PRP$. Moreover $A(PRP) \cong A(R)^{op}$ as braided bialgebras. Thus if R is a Cremmer-Gervais R-matrix, then we may construct $\mathbb{C}_{\mathbb{R}'}[\mathbb{GL}(\ltimes)]$ analogously and we will have that $\mathbb{C}_{\mathbb{R}'}[\mathbb{GL}(\ltimes)] \cong \mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)]^{\rtimes_1}$.

Theorem 5.3. *Let $R = R_n(q,p)$ and let $\bar{R} = PR_{n-1}(q,p)P$. There is a homomorphism of Hopf algebras $\psi : \mathbb{C}_{\bar{\mathbb{R}}}[\mathbb{GL}(\ltimes - \not\Vdash)] \to \mathbb{U}_{\mathbb{R}}(\mathfrak{gl}(\mathfrak{n}))$ given by*

$$\psi(\bar{T}_i^k) = l^-(T_i^k) = l^+(T_{i+1}^{k+1})$$

Moreover, $\mathrm{Im}(\psi) \subset U_0$ *and when considered as a map from $\mathbb{C}_{\bar{\mathbb{R}}}[\mathbb{GL}(\ltimes - \not\Vdash)]$ to U_0, ψ is a morphism of braided Hopf algebras.*

Proof. Set $t_i^k = l^-(T_i^k) = l^+(T_{i+1}^{k+1})$ for $1 \le i,k \le n-1$. Then $\Delta t_i^k = \sum t_i^j \otimes t_j^k$ and $(t_i^k \mid t_j^l) = \bar{R}_{ji}^{lk}$. The vector space $V = \sum \mathbb{C}\approx_{\sqsupset}^{\not\Vdash}$ is an $n-1$ dimensional U_0-comodule on which the endomorphism (3.1) induced by the braiding is given by

$$\sigma_V(t_i^1 \otimes t_j^1) = \bar{R}_{ji}^{lk}\, t_l^1 \otimes t_k^1.$$

By the universal property of $A(\bar{R})$ [21], there is a bialgebra map $\psi : A(\bar{R}) \to U_0$ such that

$$\psi(\bar{T}_i^k) = t_i^k$$

Since $\langle a \mid b\rangle = (\psi(a) \mid \psi(b))$ for all a and b in the coalgebra spanned by the $\bar{T}_i^k$, it follows by induction that ψ is a homomorphism of braided bialgebras. It remains to show that the image of the determinant element of $A(\bar{R})$ is invertible in U_0. For if so, ψ extends to a cobraided bialgebra map $\psi : \mathbb{C}_{\bar{\mathbb{R}}}[\mathbb{GL}(\ltimes - \not\Vdash)] \to \mathbb{U}_{\not\vdash}$ [13, 3.2] which must be a Hopf algebra map [20, III.8.9].

Using 5.2 we obtain

$$l^+(\det_q) = \left[\sum_{\sigma(1)=1} a_\sigma l^+(T_n^{\sigma(n)}) \dots l^+(T_2^{\sigma(2)})\right] l^+(T_1^1)$$

and

$$l^-(\det_q) = l^-(T_n^n)\left[\sum_{\sigma(n)=n} a_\sigma l^-(T_{n-1}^{\sigma(n-1)}) \dots l^-(T_1^{\sigma(1)})\right]$$

Set $d = \sum_{\sigma(1)=1} a_\sigma l^+(T_n^{\sigma(n)}) \dots l^+(T_2^{\sigma(2)})$ and notice that by 5.2,

$$d = \sum_{\sigma(n)=n} a_\sigma l^-(T_{n-1}^{\sigma(n-1)}) \dots l^-(T_1^{\sigma(1)}).$$

Thus d is an invertible element of U_0. Denote by $\overline{\det_q}$ the determinant element of $A(\bar{R})$. Then

$$\psi(\overline{\det_q}) = \sum_{\sigma \in S_{n-1}} a_\sigma l^-(T_{n-1}^{\sigma(n-1)}) \dots l^-(T_1^{\sigma(1)}) = d.$$

□

Theorem 5.4. *Let $\phi : \mathbb{C}_{\mathbb{R}}[\mathbb{GL}(\ltimes)] \to \mathbb{C}_{\bar{\mathbb{R}}}[\mathbb{GL}(\ltimes - \not\Vdash)]^\circ$ be the induced map. Then for all $1 \le i,k \le n-1$,*

$$\phi(T_{i+1}^{k+1}) = l^+(t_i^k), \qquad \phi(T_i^k) = l^-(t_i^k).$$

Hence ϕ maps onto the subalgebra $U_{\bar{R}}(\mathfrak{gl}(\mathfrak{n} - \mathfrak{1}))$.

Proof. Apply Theorem 2.2 to the case where B is the image of ψ above. Since $l^+(T_{i+1}^{k+1}) = t_i^k$, we obtain that $\phi(T_{i+1}^{k+1}) = l^+(t_i^k)$. Similarly one obtains that $\phi(T_i^k) = l^-(t_i^k)$. Since $\phi(T_1^n) = \phi(T_n^1) = 0$, the image of ϕ lies inside $U(B)$. Thus ϕ maps onto $U(B)$ which is isomorphic to $U_{\bar{R}}(\mathfrak{gl}(\mathfrak{n} - \mathfrak{1}))$ by Theorem 2.3. □

Finally, we interpret the last two results for $\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)]$ in the one parameter case. Suppose that $p^n = q$. Set $U_R(\mathfrak{sl}(\mathfrak{n})) = \mathfrak{U}(\mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)])$ and notice that since $(\det_q - 1)$ is a braided Hopf ideal, $U_R(\mathfrak{sl}(\mathfrak{n}))$ is naturally isomorphic to $U_R(\mathfrak{gl}(\mathfrak{n}))$. Notice also that in this case $\bar{R} = PR_{n-1}(q,p)P$ is *not* the one parameter matrix.

Theorem 5.5. *Suppose that $q = p^n$. Let $R = R_n(q,p)$ and let $\bar{R} = PR_{n-1}(q,p)P$. There is a homomorphism of braided Hopf algebras $\psi : \mathbb{C}_{\bar{\mathbb{R}}}[\mathbb{GL}(\ltimes - \nVdash)] \to \mathbb{U}_{\mathbb{R}}(\mathfrak{sl}(\mathfrak{n}))_{\mathfrak{o}}$. This map induces a surjective Hopf algebra homomorphism $\phi : \mathbb{C}_{\mathbb{R}}[\mathbb{SL}(\ltimes)] \to \mathbb{U}_{\bar{\mathbb{R}}}(\mathfrak{gl}(\mathfrak{n}-\mathfrak{1}))$.*

6. The dual of a factorizable Lie bialgebra

In this final section we prove a result for factorizable (quasi-triangular) Lie bialgebras which can be viewed as the 'semi-classical limit' of the results of the previous section. We prove that, although the dual of a factorizable Lie algebra $\mathfrak{g}$ is rarely itself factorizable, there is a canonical homomorphic image of $\mathfrak{g}^*$ that has a natural factorizable Lie bialgebra structure. As is often the case, the theorem in the Lie bialgebra situation is at the same time much more general and much easier to prove. We first formulate the result in such a way as to emphasize the connections with the results in Section 2. For the proof and application to reductive Lie algebras, we reformulate slightly the notation.

Recall that a quasi-triangular Lie bialgebra is a Lie bialgebra $\mathfrak{g}$ such that the cocommutator δ is of the form $\delta = dr$ where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation. Note that the axioms for δ imply that $t = r_{12} + r_{21}$ is an invariant (symmetric) element of $\mathfrak{g} \otimes \mathfrak{g}$. Following [25] we shall say that $\mathfrak{g}$ is factorizable if t is non-degenerate (as a bilinear form on $\mathfrak{g}^*$). Suppose that $r = \sum r_i \otimes r_i'$. Define $\phi_\pm : \mathfrak{g}^* \to \mathfrak{g}$ by

$$\phi_+(\xi) = \sum \xi(r_i) r_i', \quad \phi_-(\xi) = -\sum \xi(r_i') r_i$$

Then $\phi_\pm$ are Lie algebra homomorphisms whose duals are given by $\phi_\pm^* = -\phi_\mp$ [5, 26]. Recall that for any bialgebra $\mathfrak{g}$ we may define $\mathfrak{g}^{\mathfrak{op}}$ (respectively $\mathfrak{g}^{\mathfrak{cop}}$) to be $\mathfrak{g}$ equipped with the same cocommutator but the negative or opposite commutator (resp. with the same commutator but the opposite cocommutator). Thus we have Lie bialgebra maps,

$$\phi_\pm : (\mathfrak{g}^*)^{\mathfrak{cop}} \to \mathfrak{g}.$$

Let $\mathfrak{c}_\pm = \operatorname{Im} \phi_\pm$ and let $\mathfrak{c}_{\mathfrak{o}} = \mathfrak{c}_+ \cap \mathfrak{c}_-$. Then all three of $\mathfrak{c}_{\mathfrak{1}}$, $\mathfrak{c}_{\mathfrak{2}}$ and $\mathfrak{c}_{\mathfrak{o}}$ are Lie subbialgebras of $\mathfrak{g}$. Let $\mathfrak{k} = \mathfrak{c}_{\mathfrak{o}}^\perp$. Our main result is the following.

Theorem 6.1. *The Lie bialgebra $\mathfrak{g}^*/\mathfrak{k} \cong \mathfrak{c}_{\mathfrak{o}}^*$ is factorizable.*

Before proving this result, we revert to the notation of [3]. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the bilinear form. The bilinear form on $\mathfrak{g}^*$ induces an invariant bilinear form on $\mathfrak{g}$. Identify ϕ_+ with $f \in \operatorname{End} \mathfrak{g}$. Then f satisfies

$$f + f^* = 1; \tag{6.1}$$

$$[f(x), f(y)] = f([x, f(y)] + [f(x), y] - [x, y]). \tag{6.2}$$

(where f^* denote the adjoint of f). Conversely, suppose that $\mathfrak{g}$ is a Lie algebra with a fixed nondegenerate symmetric bilinear form. Then the existence of a factorizable Lie bialgebra structure on $\mathfrak{g}$ is equivalent to the existence of an operator $f \in \operatorname{End} \mathfrak{g}$ such that (6.1) and (6.2) hold. We denote the corresponding factorizable Lie

bialgebra by the pair $(\mathfrak{g}, \mathfrak{f})$. We may identify $\mathfrak{g}^*$ with $\mathfrak{g}$ equipped with a new Lie bracket given by

$$[x, y]_f = [x, f(y)] + [f(x), y] - [x, y]$$

(see [26]). Denote this Lie algebra by $\mathfrak{g}_\mathfrak{f}$. As above we have a pair of Lie bialgebra maps,

$$f, f - 1 : \mathfrak{g}_\mathfrak{f}^{\mathfrak{cop}} \to \mathfrak{g}.$$

Let $\mathfrak{c}_1 = \mathrm{Im}(\mathfrak{f} - 1)$ and note that $\mathfrak{c}_1^\perp = \mathrm{Ker}\,\mathfrak{f}$. So f induces an endomorphism $\tilde{f}$ of $\mathfrak{c}_1/\mathfrak{c}_1^\perp$ which satisfies conditions (6.1) and (6.2) above. Hence $\mathfrak{c}_1/\mathfrak{c}_1^\perp$ has a factorizable Lie bialgebra structure.

Theorem 6.2. *The map $f - 1 : \mathfrak{g}_\mathfrak{f}^{\mathfrak{cop}} \to \mathfrak{c}_1/\mathfrak{c}_1^\perp$ is a Lie bialgebra homomorphism with kernel $\mathfrak{k} = \mathrm{Ker}\,\mathfrak{f} + \mathrm{Ker}(\mathfrak{f} - 1)$.*

Proof. It is clear that $f - 1 : \mathfrak{g}_\mathfrak{f}^{\mathfrak{cop}} \to \mathfrak{c}_1/\mathfrak{c}_1^\perp$ is a Lie algebra homomorphism with kernel $\mathfrak{k} = \mathrm{Ker}\,\mathfrak{f} + \mathrm{Ker}(\mathfrak{f} - 1)$. Thus it suffices to show that the dual map $(f - 1)^* : (\mathfrak{c}_1/\mathfrak{c}_1^\perp)^* \to (\mathfrak{g}_\mathfrak{f}^{\mathfrak{cop}})^*$ is also a Lie algebra homomorphism. Set $\mathfrak{c} = \mathfrak{f}(\mathfrak{c}_1)$. Then $\mathfrak{c}^\perp = \mathfrak{k}$. Identifying $(\mathfrak{c}_1/\mathfrak{c}_1^\perp)^*$ with $(\mathfrak{c}_1/\mathfrak{c}_1^\perp)_{\tilde{\mathfrak{f}}}$ and $(\mathfrak{g}_\mathfrak{f}^{\mathfrak{cop}})^*$ with $\mathfrak{c}^{\mathfrak{op}}$ via the bilinear form, we find that $(f-1)^*$ identifies with the vector space isomorphism $-f : (\mathfrak{c}_1/\mathfrak{c}_1^\perp)_{\tilde{\mathfrak{f}}} \to \mathfrak{c}^{\mathfrak{op}}$. Since, for all $x, y \in \mathfrak{c}_1$,

$$f([\bar{x}, \bar{y}]_{\tilde{f}}) = f([x, y]_f) = [f(x), f(y)]$$

the map $-f$ above is a Lie algebra isomorphism. Hence $f - 1$ is an isomorphism of Lie bialgebras. □

Theorm 6.1 may then be deduced immediately from this result by making the appropriate identifications.

We now look more closely at the case when $\mathfrak{g}$ is a reductive complex Lie algebra. The work of Belavin and Drinfeld [3] is easily extended to give a complete description of factorizable Lie bialgebra structures. Suppose that $\mathfrak{g}$ is a reductive Lie algebra equipped with a non-degenerate invariant symmetric bilinear form κ. Let $\mathfrak{h}$ be a Cartan subalgebra, let R be the root system and choose a base B for R. Denote the corresponding positive and negative roots by R_+ and R_- respectively. An admissible triple is a triple (B_1, B_2, τ) where $B_1,\ B_2 \subset B$ and $\tau : B_1 \to B_2$ is a bijection such that

$$\text{for all } \alpha, \beta \in B_1, \quad (\tau(\alpha), \tau(\beta)) = (\alpha, \beta); \tag{6.3}$$

$$\text{for all } \alpha \in B_1 \text{ there exists a } t \text{ such that } \tau^t(\alpha) \notin B_1. \tag{6.4}$$

For all $\alpha \in R$, choose elements $e_\alpha \in \mathfrak{g}^\alpha$ such that $\kappa(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}$. Set $h_\alpha = [e_\alpha, e_{-\alpha}]$. An admissible quadruple is a pair (τ, f_0) where $f_0 \in \mathrm{End}\,\mathfrak{h}$ satisfies:

$$f_0 + f_0^* = 1 \tag{6.5}$$

$$f_0(h_\alpha) = (f_0 - 1)(h_{\tau(\alpha)}) \text{ for all } \alpha \in B_1 \tag{6.6}$$

Given any admissible quadruple, we construct an operator $f \in \mathrm{End}(\mathfrak{g})$ in the following way. First define an ordering on R_+ by $\beta \succeq \alpha$ if and only if there exists a non-negative integer j such that $\beta = \tau^j(\alpha)$. Define, for all $\alpha \in R_+$,

$$f(e_\alpha) = -\sum_{\beta \succ \alpha} e_\beta, \quad f(e_{-\alpha}) = \sum_{\beta \preceq \alpha} e_{-\beta}, \quad f|_\mathfrak{h} = f_0.$$

The proof of Belavin and Drinfeld in the simple case [3] may be easily extended to give the following generalization of their original result. (The result in the semi-simple case is stated and proved in [4]).

Theorem 6.3 (Belavin-Drinfeld)**.** *Let $\mathfrak{g}$ be a reductive complex Lie algebra equipped with a nondegenerate invariant symmetric bilinear form. For any admissible quadruple, the operator f defined above satisfies* (6.1) *and* (6.2)*.*

Conversely, suppose that $f \in \operatorname{End}\mathfrak{g}$ satisfies (6.1) *and* (6.2) *above. Then f is the operator associated to an admissible quadruple for a suitable choice of $\mathfrak{h}$ and B.*

Given this classification we can describe very precisely the structure of $(\mathfrak{c}_1/\mathfrak{c}_1^\perp, \tilde{\mathfrak{f}})$ in terms of that of $(\mathfrak{g}, \mathfrak{f})$.

Theorem 6.4. *Let (τ, f_0) be an admissible quadruple and let $(\mathfrak{g}, \mathfrak{f})$ be the associated factorizable Lie bialgebra. Then $\mathfrak{c}_1/\mathfrak{c}_1^\perp$ is a reductive Lie algebra with Cartan subalgebra $\tilde{\mathfrak{h}} = \operatorname{Im}(f_0 - 1)/\operatorname{Im}(f_0 - 1)^\perp$ and root system R_1. The operator $\tilde{f}$ is the operator associated to the admissible quadruple $(\tilde{\tau}, \tilde{f}_0)$ where $\tilde{\tau} = \tau|_{\tau^{-1}(B_1 \cap B_2)}$ and $\tilde{f}_0$ is the map induced by f_0 on $\tilde{\mathfrak{h}}$.*

Proof. It suffices to recall the description of $\mathfrak{c}_1$ and $\mathfrak{c}_1^\perp$ given in [3]. Set $\mathfrak{a} = \sum_{\alpha \in \mathfrak{B}_1} \mathbb{C}\widetilde{\alpha} \oplus \sum_{\alpha \in \mathbb{R}_{\not\mathbb{C}}} \mathfrak{g}^\alpha$; set $\hat{\mathfrak{n}} = \sum_{\alpha \in R_+ \setminus R_1} \mathfrak{g}^\alpha$. Choose $V \subset (\mathfrak{h} \cap \mathfrak{a})^\perp$ such that $V \supset V^\perp$ and $\operatorname{Im}(f_0 - 1) = (\mathfrak{h} \cap \mathfrak{a}) \oplus \mathfrak{V}$. Then one verifies easily [3, §6.4] that

$$\mathfrak{c}_1 = \mathfrak{p} + \mathfrak{V}, \quad \mathfrak{c}_1^\perp = \hat{\mathfrak{n}} + \mathfrak{V}^\perp$$

and

$$\mathfrak{c}_1/\mathfrak{c}_1^\perp \cong \mathfrak{a} \oplus \mathfrak{V}/\mathfrak{V}^\perp.$$

This provides the first assertion. The second assertion is verified by calculating $\tilde{f}$ explicitly. □

Example . Consider the 'Cremmer-Gervais' bialgebra structure on $\mathfrak{sl}(\mathfrak{n}+1)$ (that is, the bialgebra structure that occurs as the semi-classical limit of the Cremmer-Gervais R-matrix). Set $\mathfrak{g} = \mathfrak{sl}(\mathfrak{n}+1)$, $B = \{\alpha_1, \ldots, \alpha_n\}$, $B_1 = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $B_2 = \{\alpha_2, \ldots, \alpha_n\}$ and $\tau(\alpha_i) = \alpha_{i+1}$. Suppose that $\mathfrak{g}$ has the factorizable bialgebra structure associated to this triple, (there is a unique choice of f_0 in this situation). Then $\mathfrak{c}_1/\mathfrak{c}_1^\perp \cong \mathfrak{gl}(\mathfrak{n})$, and the associated triple is the map $\tilde{\tau} : \{\alpha_1, \ldots, \alpha_{n-2}\} \to \{\alpha_2, \ldots, \alpha_{n-1}\}$ given again by $\tilde{\tau}(\alpha_i) = \alpha_{i+1}$. Thus we obtain a bialgebra structure on $\mathfrak{gl}(\mathfrak{n})$ of 'Cremmer-Gervais-type' but with a non-trivial $\tilde{f}_0$. This should be compared with Theorem 5.5.

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