Determinant Test for Positive and Negative Definite Matrices

THEOREM. Let A be an $n \times n$ symmetric real matrix and let D_k be the determinant of the matrix obtained from A by deleting the last $n - k$ rows and columns of A. Then

1) A is positive definite if and only if $D_1, \dots, D_n$ are strictly positive; and

2) A is negetive definite if and only if $D_1, \dots, D_n$ alternate in sign.

PROOF 1). We prove this by induction. The case $n = 1$ is easily verified.

Now assume that the result is true for $n \times n$ matrices. Assume that A is an $(n + 1) \times (n + 1)$ matrix and assume that the determinants $D_1, \dots, D_{n+1}$ of the principal minors of A are strictly positive. Then the matrix B obtained from A by deleting the last row and column of A is positive definite. There is an orthonormal basis of eigenvectors of B, i.e., an orthonormal basis $\mathbf{x}_1, \dots, \mathbf{x}_n$ of $\Re^n$ and strictly positive numbers $\lambda_1, \dots, \lambda_n$ such that

$$B\mathbf{x}_i = \lambda_i\mathbf{x}_i \qquad (1 \le i \le n).$$

There is an orthogonal $n \times n$ matrix P such that

$$P^TBP = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n).$$

Here P is the matrix with respective columns $\mathbf{x}_i$. Then the $(n + 1) \times (n + 1)$ matrix Q given by

$$Q = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$$

has the property that

$$Q^TAQ = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & a_1 \\ 0 & \lambda_2 & \dots & 0 & a_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n & a_n \\ a_1 & a_2 & \dots & a_n & a_{n+1} \end{bmatrix}.$$

In fact we note that Q^TAQ is again symmetric. We have that

$$\mathrm{Det}(Q^TAQ) = (\mathrm{Det}\ Q)^{-1}\ (\mathrm{Det}\ A)\ (\mathrm{Det}\ Q) = \mathrm{Det}\ A = D_n > 0$$

from the rules of manipulation of determinants. We also have that a determinant does not change when we add a scalar multiple of one column to another column. In our case, this gives

$$\mathrm{Det}\begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n & 0 \\ a_1 & a_2 & \dots & a_n & a_{n+1} - \sum\limits_{i=1}^{n} (a_i^2/\lambda_i) \end{bmatrix} > 0$$

The preceding determinant is

$$\lambda_1 \dots \lambda_n \left(a_{n+1} - \sum_{i=1}^{n} \frac{a_i^2}{\lambda_i} \right).$$

Since the λ_i are strictly positive, we get that

$$a_{n+1} - \sum_{i=1}^{n} \frac{a_i^2}{\lambda_i} > 0.$$

We now show that Q^TAQ is positive definite. We have that

$\mathbf{y}^TQ^TAQ\mathbf{y}$

$$= (y_1, y_2, \dots, y_n, y_{n+1}) \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & a_1 \\ 0 & \lambda_2 & \dots & 0 & a_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n & a_n \\ a_1 & a_2 & \dots & a_n & a_{n+1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \\ y_{n+1} \end{bmatrix}$$

$$= \left(\lambda_1 y_1 + a_1 y_{n+1}, \lambda_2 y_2 + a_2 y_{n+1}, \dots, \lambda_n y_n + a_n y_{n+1}, \sum_{i=1}^{n} a_i y_i\right) \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \\ y_{n+1} \end{bmatrix}$$

$$= (\lambda_1 y_1 + a_1 y_{n+1}) y_1 + \dots + (\lambda_n y_n + a_n y_{n+1}) y_n + \left(\sum_{i=1}^{n} a_i y_i\right) y_{n+1}$$

$$= \sum_{i=1}^{n} \lambda_i y_i^2 + \sum_{i=1}^{n} 2\, a_i y_i y_{n+1} + a_{n+1} y_{n+1}^2$$

$$= \sum_{i=1}^{n} \left(\lambda_i a_i^2 + 2\, a_i y_i y_{n+1} + \frac{a_i^2}{\lambda_i} y_{n+1}^2\right) + \left(a_{n+1} - \sum_{i=1}^{n} \frac{a_i^2}{\lambda_i}\right) y_{n+1}^2$$

$$= \sum_{i=1}^{n} \left(\lambda_i^{1/2} y_i + \frac{a_i}{\lambda_i^{1/2}} y_{n+1}\right)^2 + \left(a_{n+1} + \sum_{i=1}^{n} \frac{a_i^2}{\lambda_i}\right) y_{n+1}^2.$$

Now we see that $\mathbf{y} \neq 0$ impλies that

$$\mathbf{y}^TQ^TAQ\mathbf{y} > 0.$$

In fact, if $y_{n+1} \neq 0$, this is clear from the preceding expression. In $y_{n+1} = 0$, then we get that

$$\mathbf{y}^TQ^TAQ\mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2 > 0.$$

So we have that

$$\mathbf{z}^TA\mathbf{z} = (\mathbf{z}^TQ)Q^TAQ(Q^T\mathbf{z}) > 0$$

for every $\mathbf{z} \neq 0$. The induction step is now complete.

The converse of part 1 is clear. In fact, suppose A is a positive definite $n \times n$ matrix and that $1 \leq m < n$; then

$$(x_1, \dots, x_m, 0, \dots, 0)^T A (x_1, \dots, x_m, 0, \dots, 0) > 0$$

for every nonzero vector $(x_1, \dots, x_m, 0, \dots, 0)$. This means that the $m \times m$ submatrix of A consisting of the first m rows and columns of A is positive definite and thus has a strictly positive determinant.

PROOF 2). The matrix C is negative definite matrix if and only if the matrix A = - C is positive definite. We have that A is positive definite if and only if $D_1, \ldots, D_n$ are strictly positive where the D_i are the determinants of the principal minors of A. But we have that

$$D_i = (-1)^i \text{Det } C_i$$

where C_i is the i^{th} principal minor of C. This gives the statement in (2).

COROLLARY. Let $A = (a_{ij})$ be a symmetric 2×2 matrix. Then

1) A is positive definite if $a_{11} > 0$ and $a_{11}a_{22} - (a_{12})^2 > 0$;

2) A is negative definite if $a_{11} < 0$ and $a_{11}a_{22} - (a_{12})^2 > 0$; and

3) there are nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\neg^2$ with $\mathbf{v}_1^T A \mathbf{v}_1 > 0$ and $\mathbf{v}_2^T A \mathbf{v}_2 < 0$ if $a_{11}a_{22} - (a_{12})^2 < 0$.

PROOF. Part (3) states that A is nonsingular but neither positive or negative definite, viz., A has eigenvalues λ_i with $\lambda_1\lambda_2 < 0$.