In the second edition of James Stewart's *Calculus - Concepts and Contexts*, Problem 13 of page 621 asks for the Taylor series associated with the function $f(x) = \sin x$ about $a = \pi/4$.

Of course, we are to specialize the formula

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n \tag{1}$$

to the situation where $f(x) = \sin x$ and $a = \pi/4$.

Here is a more interesting solution than the one suggested by the answer given there.

The first derivative of $f(x)$ is $f^{(1)}(x) = \cos x = \sin\bigl(x + \frac{\pi}{2}\bigr)$.

The second derivative of $f(x)$ is $f^{(2)}(x) = -\sin x = \sin\bigl(x + 2\,\frac{\pi}{2}\bigr)$.

The third derivative of $f(x)$ is $f^{(3)}(x) = -\cos x = \sin\bigl(x + 3\,\frac{\pi}{2}\bigr)$.

The fourth derivative of $f(x)$ is $f^{(4)}(x) = \sin x = \sin\bigl(x + 4\,\frac{\pi}{2}\bigr)$.

The fifth derivative of $f(x)$ is $f^{(5)}(x) = \cos x = \sin\bigl(x + 5\,\frac{\pi}{2}\bigr)$.

Mathematical induction establishes the validity of the formula

$$f^{(n)}(x) = \sin\bigl(x + n\,\frac{\pi}{2}\bigr), \quad \text{for } n = 0,\ 1,\ 2,\ \ldots \text{ and any } x.$$

Thus, for any a, the corresponding Taylor series (1) about a for $\sin x$ is

$$\sum_{n=0}^{\infty} \frac{\sin\bigl(a + n\,\frac{\pi}{2}\bigr)}{n!}\,(x-a)^n \tag{2}$$

The interesting feature of Problem 13 is the further simplification when $a = \pi/4$. Namely, we then have

$$\sin\bigl(\frac{\pi}{4} + n\,\frac{\pi}{2}\bigr) = \frac{1}{\sqrt{2}}\cos\bigl(n\,\frac{\pi}{2}\bigr) + \frac{1}{\sqrt{2}}\sin\bigl(n\,\frac{\pi}{2}\bigr) = \frac{1}{\sqrt{2}}\,h(n), \quad \text{for } n = 0,\ 1,\ 2,\ \ldots, \tag{3}$$

where $h(n) = \sin\bigl(n\,\frac{\pi}{2}\bigr) + \cos\bigl(n\,\frac{\pi}{2}\bigr)$. We find that:

$$\begin{aligned}
h(n) &= 1, \quad \text{when } n = 4k \text{ and } k = 0,\ 1,\ 2,\ \ldots;\\
h(n) &= 1, \quad \text{when } n = 4k+1 \text{ and } k = 0,\ 1,\ 2,\ \ldots;\\
h(n) &= -1, \quad \text{when } n = 4k+2 \text{ and } k = 0,\ 1,\ 2,\ \ldots; \text{ and}\\
h(n) &= -1, \quad \text{when } n = 4k+3 \text{ and } k = 0,\ 1,\ 2,\ \ldots.
\end{aligned}$$

By considering each of the four cases just mentioned, we discovered through trial and error that

$$h(n) = (-1)^{\frac{n(n-1)}{2}}, \quad \text{for any nonnegative integer } n. \tag{4}$$

Consequently, we see that (2), (3), and (4) yield

$$\sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{\sqrt{2}\,n!}\,(x - \frac{\pi}{2})^n \tag{5}$$

for the Taylor series of $\sin x$ about $a = \pi/4$. We prefer the form of our answer in (5) to that of our textbook.
