A Correct Simplification for Arcsin(sin(x))

For any real number x, let Floor(x) be the greatest integer that is less than or equal to x. For instance, we have Floor(7) = 7, Floor(23/5) = 4, Floor(e) = 2, and $Floor(\pi) = 3$.

PROPOSITION. *The functions defined on the real line by*

$$f(x) = \arcsin(\sin x) \quad \text{and} \quad g(x) = \left[x - \pi\,\mathrm{Floor}\left(\frac{x}{\pi} + \frac{1}{2}\right)\right](-1)^{\mathrm{Floor}\left(\frac{x}{\pi}+\frac{1}{2}\right)} \tag{1}$$

satisfy $f(x) = g(x)$, for all real numbers x.

PROOF. Let x be a fixed real number. Then there is a unique integer n such that

$$-\frac{\pi}{2} \le x - n\pi < \frac{\pi}{2}.$$

Since the preceding statement is equivalent to

$$n \le \frac{x}{\pi} + \frac{1}{2} < n + 1,$$

we sees that the integer n is given by $n = \mathrm{Floor}\left(\frac{x}{\pi} + \frac{1}{2}\right)$.

There are two cases depending on whether n is an even integer or an odd integer.

Case 1. Suppose that $n = 2k$. With $\arcsin(\sin u) = u$ when $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$, we have

$$\sin x = \sin(x - 2k\pi) = \sin(x - n\pi)$$

and

$$\arcsin(\sin x) = \arcsin\big(\sin(x - n\pi)\big) = x - n\pi = \Big[x - n\pi\Big](-1)^n. \tag{2}$$

Case 2. Suppose that $n = 2k + 1$. After observing that

$$\begin{aligned}\sin x = \sin(x - 2k\pi) &= \sin\big(x - (2k+1)\pi + \pi\big)\\ &= \Big(\sin\big(x - (2k+1)\pi\big)\Big)\cos\pi + \Big(\cos\big(x - (2k+1)\pi\big)\Big)\sin\pi\\ &= -\sin\big(x - (2k+1)\pi\big),\end{aligned}$$

we use $\arcsin(\sin u) = u$, when $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$, to obtain

$$\begin{aligned}\arcsin(\sin x) &= \arcsin\Big(-\sin\big(x - (2k+1)\pi\big)\Big) = -\arcsin\Big(\sin\big(x - (2k+1)\pi\big)\Big)\\ &= -\arcsin\big(\sin(x - n\pi)\big) = -(x - n\pi) = \Big[x - n\pi\Big](-1)^n.\end{aligned} \tag{3}$$

With $n = \mathrm{Floor}\left(\frac{x}{\pi} + \frac{1}{2}\right)$, we combine (2) and (3) to obtain $\arcsin(\sin x) = g(x)$, where $g(x)$ is given in (1). Thus, we see that $f(x) = g(x)$ for any real number x. This completes the proof. □

The preceding result yields

$$x - \arcsin(\sin x) \equiv x - \left[x - \pi\,\mathrm{Floor}\left(\frac{x}{\pi} + \frac{1}{2}\right)\right](-1)^{\mathrm{Floor}\left(\frac{x}{\pi}+\frac{1}{2}\right)}, \quad \text{for all real } x.$$

The right member of the preceding identity is a continuous nondecreasing function of x whose graph consists of line segments over intervals of length π with slopes that are alternately 0 and 2.