Write your answers in the Blue Book. Print your names & write the number of students taking this test in the upper right corner of the Blue Book. Put this test & the Blue Book in the provided envelope.

1. The cube shown below has each side of length 2. Let $\mathcal{P}$ be the midpoint of its edge. Find the area and the perimeter of the shaded triangle, or gray triangle.

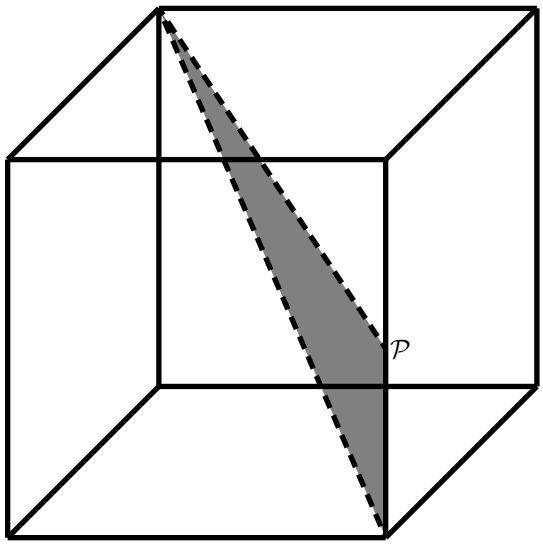


Figure 1: What is the area & the perimeter of the gray triangle?

**Solution:** Label the vertices of the cube so that the top face has vertices $A, B, C, D$ going clockwise from the upper left corner, and the bottom face has vertices $E, F, G, H$ going clockwise from the lower left corner. Then $P$ is the midpoint of the edge from vertex $C$ to vertex $G$. The problem is to find the area and the perimeter of triangle $APG$. The length of the top diagonal $\overline{AC}$ is $2\sqrt{2}$. Then $AP$ is the hypotenuse of triangle $ACP$, so $\overline{AP} = \sqrt{(2\sqrt{2})^2 + 1^2} = \sqrt{9} = 3$, and $AG$ is the hypotenuse of triangle $ACG$, so $\overline{AG} = \sqrt{(2\sqrt{2})^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$. The perimeter of triangle $APG$ is $\overline{AP} + \overline{PG} + \overline{AG} = 3 + 1 + 2\sqrt{3} = 4 + 2\sqrt{3}$. Triangle $APG$ can be seen to have height $\overline{AC}$ and base $\overline{PG}$, so its area is $\frac{1}{2}(1)(2\sqrt{2}) = \sqrt{2}$.

2. If the numerator of a fraction is selected from $\{1, 3, 5, 7, 9\}$ and the denominator of the fraction is selected from $\{2, 3, 4, 5, 6, 7\}$, what is the probability that the resulting fraction can be written as a terminating decimal? Express your answer as a common fraction.

**Solution:** If the numerator of a fraction is selected from $\{1, 3, 5, 7, 9\}$ and the denominator of the fraction is selected from $\{2, 3, 4, 5, 6, 7\}$, the resulting fraction will be a terminating decimal as long as, when the fraction is written in lowest terms, the prime divisors of the denominator consist solely of 2's and 5's. In this case that means in lowest terms the denominator needs to be $2, 4, or 5$. There are a total of $5 \cdot 6 = 30$ possibilities for fractions before they are reduced. Fifteen of those have

denominator $2, 4, 5$. Of the remaining 15 (with denominators $3, 6, 7$), they will be terminating decimals if and only if the numerator is a multiple of 3, in the case of denominators 3 and 6, or if the numerator is 7 in the case of denominator 7. That means the terminating decimal options are the five fractions $3/3, 3/6, 7/7, 9/3.9/6$. Thus a total of 20 of the 30 options, or $2/3$ will have expressions as terminating decimals.

3. On 3/7/2009, a group of talented math students was asked to determine the remainder of the number $3 \cdot 7^{2009}$ after division by 100. Naturally they got the correct solution. What was their answer?

**Solution:** Determining the remainder of the number $3 \cdot 7^{2009}$ after division by 100 means finding the units and ten digits of this number. If we begin considering powers of 7, we have $7^2 = 49, 7^3 = 343, 7^4 = 2401$. Then for any positive integer $k$, $7^{4k}$ will be a number that ends in the digits 01. Since $2008 = 4 \cdot 502$, it follows that $7^{2008}$ is a digit that ends in 01, in other words, $7^{2008} = n \cdot 100 + 1$ for some integer $n$. Then $3 \cdot 7^{2009} = 21 \cdot 7^{2008}$, so this number will end with 21.

4. A square region with perimeter 52 inches is made up out of one-inch square tiles. One tile is removed from the square, and the remaining tiles are rearranged to make a rectangle with no overlapping tiles. What is the smallest possible perimeter for such a rectangle?

**Solution:** A square region with perimeter 52 inches is made up out of one-inch square tiles. Therefore the length of each side must be 13, and the region must be made up of $13^2 = 169$ tiles. If one tile is removed, there are $168 = 13^2 - 1 = (13 - 1)(13 + 1)$ tiles, so one possible rectangle is $12 \times 14$, with perimeter again 52. Any rectangle with this many tiles would have to have sides of length $n$ and $168/n$, where $n$ is some divisor of 168, and its perimeter would be $2n + 336/n$. If $n$ is the larger of the two divisors, it is clear that $n < 26$ in the smallest such rectangle, since we already have found one with perimeter 52. Likewise it is clear that the smaller of the two divisors has to be larger than 6, since $336/6 = 56$. The divisors pairs of 168 that fit these conditions are $(7, 24), (8, 21), (12, 14)$. The corresponding rectangles have perimeters $62, 58$, and 52 respectively, so the rectangle with smallest perimeter is $12 \times 14$.

5. A hiker went out for a hike. The first part of his trip was on level ground, and he hiked at a speed of 4 mph. The second part was uphill, and he slowed to 3 mph. Upon reaching the top, he turned around and went back downhill at a speed of 6 mph. When he reached level ground again, he finished the trip at 4 mph. The total trip took 5 hours. How far did he hike?

**Solution:** Let $x$ be the distance the hiker hikes each direction on level ground, and $y$ be the distance he hikes uphill, and again the distance he hikes downhill. Then the first part of his trip takes $\frac{x}{4}$ hours, the uphill part takes $\frac{y}{3}$ hours, the downhill part takes $\frac{y}{6}$ hours, and the return on level ground takes $\frac{x}{4}$ hours. All together, his trip takes him $\frac{x}{2} + \frac{y}{2} = \frac{x+y}{2} = 5$ hours, so $x + y = 10$. Since he hiked that distance each direction, all together he hiked 20 miles.