Instructions: Write your answers in the blue book. Remember that you must explain your solutions. Even correct answers without complete justifications may receive little credit. Also, even if you can't completely solve a problem, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work. Have Fun!

6. Archimedes, Euclid, Fermat, and Gauss had a mathematics competition. Archimedes said "I did not finish 1st or 4th." Euclid said "I did not finish 4th." Fermat said "I finished 1st." Gauss said "I finished 4th."

   There were no ties in the competition and exactly three of the mathematicians were telling the truth about where the finished. Who was 1st and who was 4th? Justify your answers!

> **Solution:** If A is telling the lie, then A finished 1st or 4th, in which case either F or G also told a lie. Since only one lie was told, this is impossible. Therefore A was telling the truth.
>
> If E is lying then he finished 4th so that G's statement would be false as well. Therefore E is telling the truth.
>
> Because A and E are truthful, either F or G finished 4th. If F finished 4th then F and G are lying, so it must be that G finished 4th and was telling the truth.
>
> So F is telling a lie and did not finish 1st. Therefore, E finished 1st.
>
> That is: G was 4th, E was 1st, and F told the lie.

7. From the four numbers $x_1 < x_2 < x_3 < x_4$ one can select a pair in 6 ways. Suppose that each pair has a different sum and that the 4 smallest sums are 1,2,3, and 4. What are the possible values of x_4?

> **Solution:** Of the sums of pairs we can observe that $x_1+x_2 < x_1+x_3 < x_2+x_4 < x_3+x_4$. The remaining sums, $x_1 + x_4$ and $x_2 + x_3$ lie between $x_1 + x_3$ and $x_2 + x_4$ but we cannot determine their order with the information we've been given.
>
> We must have that $x_1 + x_2 = 1$ while $x_1 + x_3 = 2$. Of the 4 remaining possible sums the largest are $x_3 + x_4$ and $x_2 + x_4$. This tells us that $x_1 + x_4$ and $x_2 + x_3$ (in some order) must be the values 3 and 4.
>
> If $x_1 + x_4 = 3$ then we have
>
> $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_3 &= 2 \\ x_1 + x_4 &= 3 \\ x_2 + x_3 &= 4. \end{aligned}$$
>
> In which case $x_1 = -1/2$, $x_2 = 3/2$, $x_3 = 5/2$, and $x_4 = 7/2$.

Otherwise $x_1 + x_4 = 4$ and

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_3 &= 2 \\ x_1 + x_4 &= 4 \\ x_2 + x_3 &= 3, \end{aligned}$$

which has, as its solution, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, and $x_4 = 4$.

So the possible values of x_4 are $7/2$ and 4.

8. Define the function f for $n = 3, 4, \ldots$ by

$$f(n) = \log_2(3) \times \log_3(4) \times \cdots \times \log_{n-1}(n).$$

What is the value of $\sum_{2 \le k \le 10} f(2^k)$?

Solution: We can use the change of base formula for logarithms, $\log_b(a) = \log_c(a)/\log_c(b)$ to rewrite the definition of f in terms of $\log_2$:

$$f(n) = \log_2(3) \times \frac{\log_2(4)}{\log_2(3)} \times \frac{\log_2(5)}{\log_2(4)} \times \cdots \times \frac{\log_2(n)}{\log_2(n-1)} = \log_2(n).$$

Then

$$\sum_{2 \le k \le 10} f(2^k) = \sum_{2 \le k \le 10} \log_2(2^k) = \sum_{2 \le k \le 10} k = 54.$$

9. If $y = x^4 + x$ what is $\frac{dx}{d(y^2)}$ when $x = 1$?

Solution: Recall what this question means: we know how x and y are related. We suppose that the quantity y^2 undergoes a small change of size Δ. Then we expect that x will also undergo a small change and that the magnitude of this change will be proportional to Δ, at least when Δ is very small. The constant of proportionality is the derivative we seek.

If we write $u = y^2$ for a moment then the chain rule tells us that

$$dx/du \times du/dy = dx/dy.$$

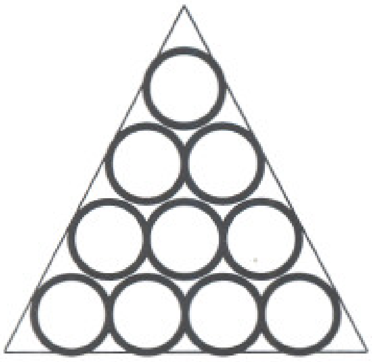
We can calculate $dx/dy = 1/(dy/dx) = 1/(4x^3 + 1)$ or, what amounts to the same thing, find dx/dy by implicit differentiation. And $du/dy = 2y$. Therefore

$$\frac{dx}{du} = \frac{1}{(4x^3 + 1)(2y)}.$$

When $x = 1$ we see that $y = 2$ and so

$$\frac{dx}{dy^2} = \frac{1}{20}.$$

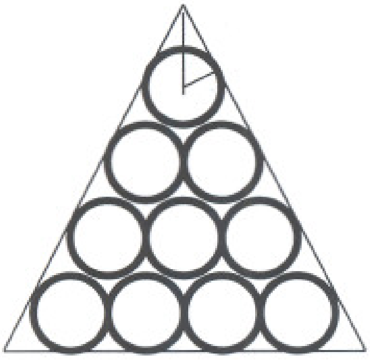
10. An equilateral triangle is completely filled with n rows of congruent, tangent circles. The case $n = 4$ is shown in the figure:



What is the value of the limit

$$\lim_{n\to\infty} \frac{\text{area of circles}}{\text{area of triangle}}$$

Solution: For a minute, take the side of the triangle to be 1 and call the radius of the circles r. Then $2(n-1)r + 2r\sqrt{3} = 1$ as seen by considering the 30-60-90 triangle whose hypotenuse connects a vertex of the triangle to the center of the nearest circle.



Hence, the area of each of the circles is $\pi/(4(n+\sqrt{3}-1)^2)$ and there are $n(n+1)/2$ circles so the area of the circles is

$$\frac{\pi}{8}\frac{n(n+1)}{(n+\sqrt{3}-1)^2}.$$

This tends to $\pi/8$ as $n \to \infty$.

Since the area of the triangle is $\sqrt{3}/4$, the ratio requested is $\pi\sqrt{3}/6$.