UC Math Contest 7 March 2009

Advanced Questions #1-5

Instructions: Write your answers in the blue book. Remember that you must explain your solutions. Even correct answers without complete justifications may receive little credit. Also, even if you can't completely solve a problem, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work. Have Fun!

1. Joe, a graffiti artist, has 2000 cans of spray paint. Explain why one of these statements *must* be true:

   1. There are at least 45 cans of the same color paint.
   2. There are at least 45 cans all of different colors of paint

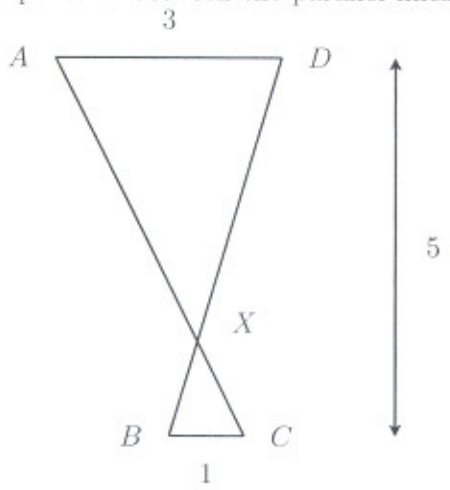
2. Define a sequence $\{a_n\}$ by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \begin{cases} 0 & \text{if } a_n = 0 \text{ and } n \text{ is odd} \\ 2 & \text{if } a_n = 0 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 1 \text{ and } n \text{ is odd} \\ 0 & \text{if } a_n = 1 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 2 \end{cases}$$

   How many of the numbers $a_1, a_2, a_3, \ldots, a_{100}$ are equal to 2?

3. Draw the graph of the points (x, y) that satisfy the equation

$$x^2 - xy + x - y = 0.$$

4. Two different, positive numbers n and m differ from their reciprocals by 1. What is the sum $m + n$?

5. Suppose that $\overline{AD}$ and $\overline{BC}$ are parallel segments with lengths 3 and 1 respectively and that the distance between them is 5. As shown in the figure, the segments $\overline{AC}$ and $\overline{BD}$ meet at the point X between the parallel lines. What is the sum of the areas of $\triangle AXD$ and $\triangle BXC$?