UC Math Contest 7 March 2009

Advanced Questions #1-5

Instructions: Write your answers in the blue book. Remember that you must explain your solutions. Even correct answers without complete justifications may receive little credit. Also, even if you can't completely solve a problem, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work. Have Fun!

1. Joe, a graffiti artist, has 2000 cans of spray paint. Explain why one of these statements *must* be true:

   1. There are at least 45 cans of the same color paint.
   2. There are at least 45 cans all of different colors of paint

Solution: If there are at most 44 colors and 44 cans of each color then there are at most $44 \times 44 = 1936$ cans in Joe's collection. This can't be the case because he has 2000 cans of paint.

In terms of symbolic logic our argument relies on the fact that "not ((not P) and (not Q))" is equivalent to "P or Q".

2. Define a sequence $\{a_n\}$ by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \begin{cases} 0 & \text{if } a_n = 0 \text{ and } n \text{ is odd} \\ 2 & \text{if } a_n = 0 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 1 \text{ and } n \text{ is odd} \\ 0 & \text{if } a_n = 1 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 2 \end{cases}$$

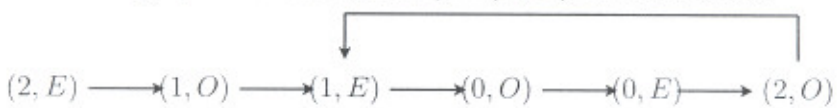
How many of the numbers $a_1, a_2, a_3, \ldots, a_{100}$ are equal to 2?

Solution: Starting with $a_1 = 1$ we find that $a_2 = 1$, $a_3 = 0$, $a_4 = 0$, $a_5 = 2$, and $a_6 = 1$. At this point the sequence repeats itself as 6 is even and $a_6 = 1$ just as 2 is even and $a_2 = 1$.

We can state the resulting pattern by saying for any positive integer k, $a_{4k} = 0$, $a_{4k+1} = 2$, $a_{4k+2} = 1$, and $a_{4k+3} = 0$. Therefore $a_n = 2$ exactly when $n = 4k + 1$. That is, $n = 5, 9, 13, \ldots, 97$ corresponding to the values $k = 1, 2, 3, \ldots, 24$.

Thus there are 24 of the numbers $\{a_n, 1 \leq n \leq 100\}$ that equal 2.

An interesting way to visualize the scheme that defines the sequence $\{a_n\}$ is to think of it as a directed graph in the set $\{0, 1, 2\} \times \{O, E\}$ as shown here:



This picture makes it clear that, starting with a_4 the values of a_n repeat in a cycle of 4.

3. Draw the graph of the points (x, y) that satisfy the equation

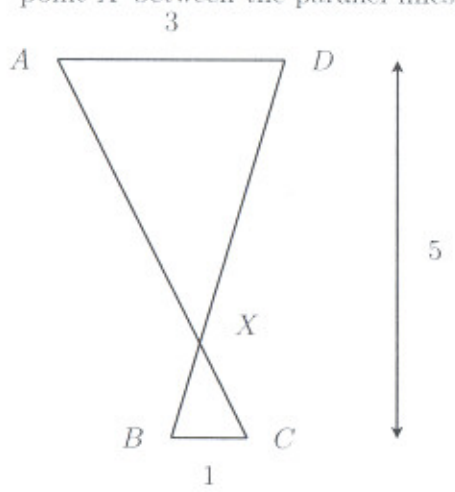
$$x^2 - xy + x - y = 0.$$

Solution: We can write $x^2 - xy + x - y = (x - y)(x + 1)$ and this product vanishes exactly when one or the other of its factors vanishes. The locus in question is therefore made up of two lines, one with the equation $y = x$ and the other with the equation $x = -1$.

4. Two different, positive numbers n and m differ from their reciprocals by 1. What is the sum $m + n$?

Solution: We're told that m and n are solutions of $|x - 1/x| = 1$. If $x > 1$ this says $x - 1/x = 1$ so that $x^2 - x - 1 = 0$. All solutions of this equation are $x = (1 \pm \sqrt{5})/2$ so a possible value of m is $(1+\sqrt{5})/2$. On the other hand, if $x < 1$ our condition says $1/x - x = 1$, or, $x^2 + x - 1 = 0$. The solutions of this quadratic equation are $x = (-1 \pm \sqrt{5})/2$. Thus the other possible value for m is $(\sqrt{5} - 1)/2$. So $m + n = \sqrt{5}$.

5. Suppose that $\overline{AD}$ and $\overline{BC}$ are parallel segments with lengths 3 and 1 respectively and that the distance between them is 5. As shown in the figure, the segments $\overline{AC}$ and $\overline{BD}$ meet at the point X between the parallel lines. What is the sum of the areas of $\triangle AXD$ and $\triangle BXC$?



Solution: The triangles in the figure are similar and the ratio of the lengths of their sides is 3. Therefore the ratio of the height of $\triangle AXD$ above its base $\overline{AD}$ is 3 times the height of $\triangle BXC$ above the base $\overline{BC}$. Since the sum of the heights is 5 the individual heights must be 15/4 and 5/4. Therefore the sum of the areas of the triangles is

$$S = \frac{1}{2} \cdot 3 \cdot \frac{15}{4} + \frac{1}{2} \cdot 1 \cdot \frac{5}{4} = \frac{25}{4}.$$