Instructions: Write your answers in the blue book. Remember that you must explain your solutions. Even correct answers without complete justifications may receive little credit. Also, even if you can't completely solve a problem, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

Have Fun!

6. Ben has over 100 pieces of candy. If he tries to divide them evenly among the three kids in his appartment building, there'll be 2 left over. If he divides the candies evenly among his 4 nieces, there'll be one left over. And, if he divides the candies among his 5 classmates, there'll be 3 left over. What's the smallest number of pieces of candy Ben could have?

Solution: We're asked to find the smallest integer $n > 100$ so that

$$\begin{aligned} n &= 2 \pmod 3 \\ n &= 1 \pmod 4 \\ n &= 3 \pmod 5 \end{aligned}$$

From the last condition we can write $n = 5k + 3$ for some integer k and therefore

$$5k + 3 = 1 \pmod 4,$$

so $k = 2 \pmod 4$ and $k = 4l + 2$ for some integer l. This tells us that $n = (4l + 2)5 + 3 = 20l + 13$. And, as $20l + 13 = 2 \pmod 3$, we must have that $l = 3m + 2$ for some integer m.

We conclude that there is an integer m so that

$$n = 20(3m + 2) + 13 = 60m + 53.$$

The smallest such n larger than 100 is obtained with $m = 1$. Then $n = 113$.

7. This is a question about the composition of functions. Let $f_0(x) = \frac{x}{x+1}$ and define, for $n \geq 0$, the functions $f_n(x)$ by

$$f_{n+1}(x) = f_0 \circ f_n(x).$$

Find a formula for $f_n(x)$.

Solution: If you compute the first few functions f_n explicitly you'll rapisly arrive at the conjecture that

$$f_n(x) = \frac{x}{(n+1)x + 1}.$$

To show that this is in fact the case the appropriate tool is mathematical induction.

Note first that the proposed formula is correct is $n = 0$. Then, as the second step in our proof, we assume the formula is correct for some integer n, and compute

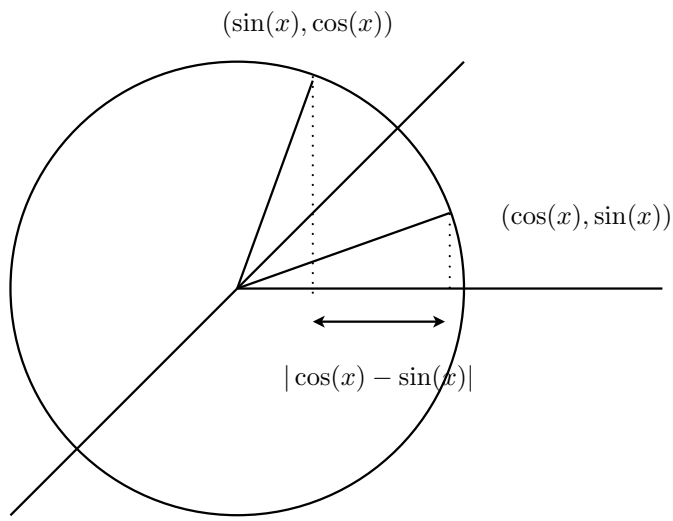
$$\begin{aligned} f_{n+1}(x) &= f_0 \circ f_n(x) \\ &= f_0\left(\frac{x}{(n+1)x+1}\right) \\ &= \frac{\frac{x}{(n+1)x+1})}{1 + \frac{x}{(n+1)x+1})} \end{aligned}$$

$$= \frac{x}{(n+2)x+1},$$

showing that the formula is correct of the next integer as well.

8. Show that for every number x, $|\cos(x) - \sin(x)| \leq \sqrt{2}$.

Solution: The expression in question represents the distance between the x coordinate of a point on the unit circle and the x-coordiante of its reflection in the line $y = x$. This is illustrated in the figure.



Evidently this distance is largest when the angle between the point and its reflection is 180 degrees. So the *largest* difference in the x coordinates occrs when $x = -\pi/4$ (or $x = 3\pi/4$) and the value of the distance then is $2\cos(\pi/4) = \sqrt{2}$.

9. The number a leaves a remainder of 2 when divided by 3. What is the remainder of a^{2007} when it is divided by 3?

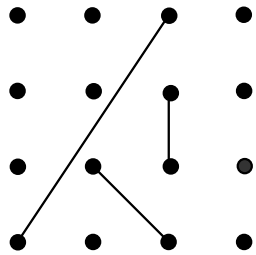
Solution: We're told that $a = 3k + 2$ so that $a^2 = (3k+2)^2 = (9k + 12 + 1)k + 1$, so that a^2 leaves a remainder of 1 when divided by 3.

If n is any positive power we can expand $(3k+1)^n$ and see that every term contains a factor of 3 except the "constant" term without any k's in it. This constant term is 1 and so b^n leaves a remainder of 1 when divided by 3 whenever b does.

$$\begin{aligned} a^{2007} &= a \cdot (a^2)^{1003} \\ &= (3k+2)(3M+1) \\ &= 9kM + 6M + 3k + 2 \\ &= 3(3kM + 2M + k) + 2. \end{aligned}$$

Thus, a^{2007} leaves a remainder of 2 when divided by 3.

10. The figure shows a 4×4 square grid of equally spaced points spaced 1 unit apart along with several line segements obtained by connecting points of the grid.

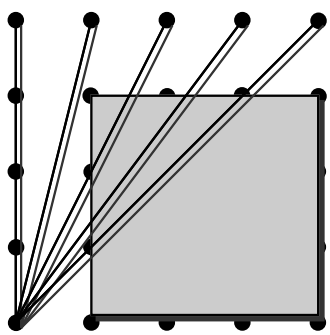


The line segments have different lengths — one of them is 1 unit long, for example, and another is $\sqrt{2}$ units long.

Find a fomula $f(n)$ that tells how many *different* numbers occur as the lengths of segements in an $n \times n$ square grid.

Solution: In a 2×2 grid thate are only two possible lengths, 1 and $\sqrt{2}$ since every segment is either a side or diagonal of the sqaure. So $f(2) = 2$.

As shown in this figure,



every length that occurs in an $n \times n$ grid also occurs in an $n+1 \times n+1$ grid, and in addition, there are n additional lengths.

Hence $f(n) = n + f(n-1)$ and so

$$f(n) = 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2} - 1.$$