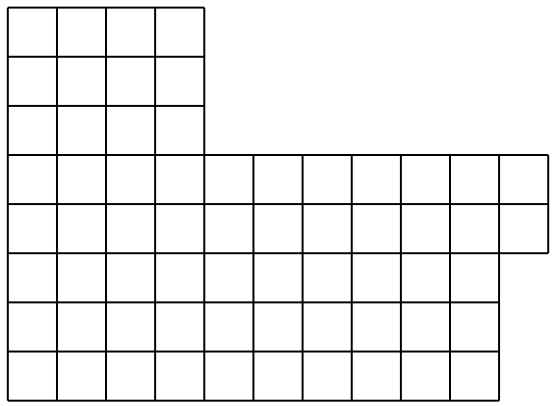
Instructions: Write your answers in the blue book. Remember that you must explain your solutions. Even correct answers without complete justifications may receive little credit. Also, even if you can't completely solve a problem, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

Have Fun!

1. Travelers stranded in an airport by a giant snow storm wanted to pass the time playing checkers. They had coins for the pieces but no board on which to play. Eventually, one of them found a piece of linoleum as shown in the illustration, and, as it contained the correct number of squares, it was decided to cut it into pieces and fit them together to make the required 8×8 board, blackening some squares afterward for convenience in playing.

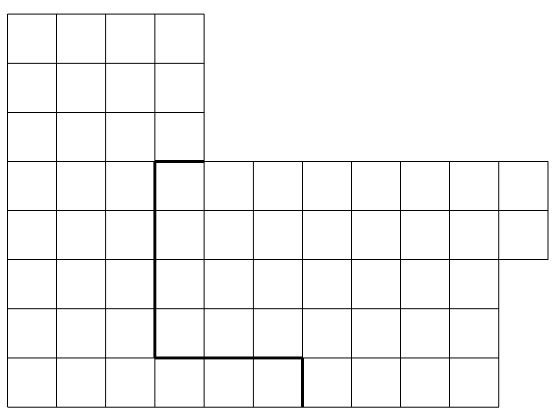


An exceptionally clever woman showed how this could be accomplished by cutting the linoleum into only two pieces. It is an interesting challenge to discover her solution.

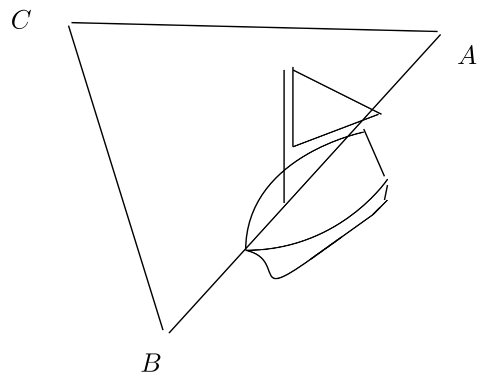
Show how to cut the linoleum along lines into two pieces that will fit together to form an 8×8 board.

Solution:

Cut the linoleum as indicated by the heavy line and then rotate the piece on the right of the figure 90 deegrees clockwise.



2. In the sketch below, the ship is on the first leg of a race on a triangular course from A to B to C and back to A. Three landlubbers on the ship tried to keep track of the ship's time but were terribly seasick and their observations suffered accordingly.



Smith observed that the ship did the first three-quarters of the route in three and a half hours. Jones noted only that it did the final three-quarters of the route in four and a half hours. And poor Williams only observed that the second leg of the route (from B to C) took ten minutes more than the first leg. Assuming that the points A, B, and C form an equilateral triangle and that the ship's speed was constant on each leg of the route, how long did it take the ship to complete the race? (over)

Solution: The first leg was sailed in 80 minutes, the second in 90 and the last in 160 make a total time of 5.5 hours.

For an algebraic solution, divide the course into 12 equal parts, 4 on each side of the triangle. Let x and y be the times for the first and last legs of the route. Then, converting everything to minutes, the data given tell us

$$\begin{aligned} \frac{x}{4} + x + 10 + y &= 270 \\ \frac{y}{4} + x + 10 + x &= 210 \end{aligned}$$

and this system is easy to solve.

3. The graph of quadratic polynomial $p(x)$ passes through the points $(0,0)$, $(1,1)$, and $(2,0)$. Find all numbers T so that

$$\int_0^T p(s)\,ds = 0.$$

Solution: The polynomial p is $2x - x^2$. One way to see this is assume that $p(x) = a + bx + cx^2$ and solve the equations

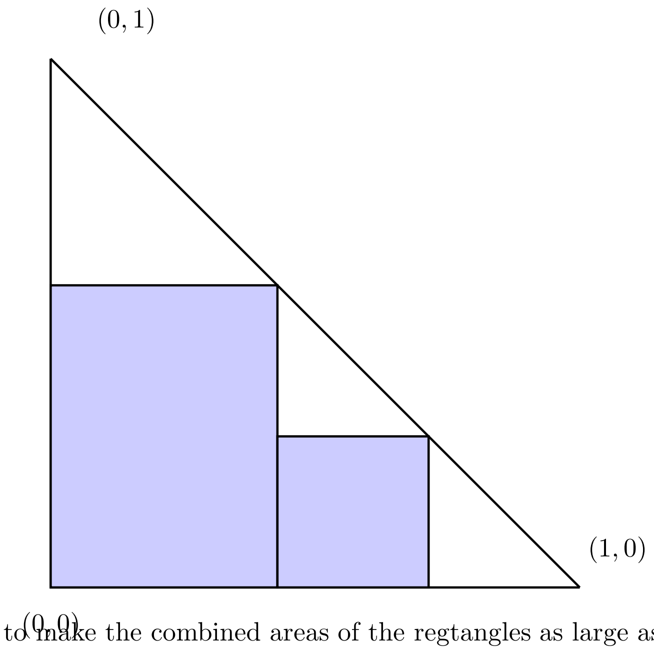
$$\begin{aligned} p(0) &= 0 \\ p(1) &= 1 \\ p(2) &= 0 \end{aligned}$$

for a, b, c.

Then $\int_0^T p(s)\,ds = T^2 - T^3/3 = T^2(1 - T/3)$.

So the solutions of $\int_0^T p(s)\,ds = 0$ are $T = 0$ and $T = 3$.

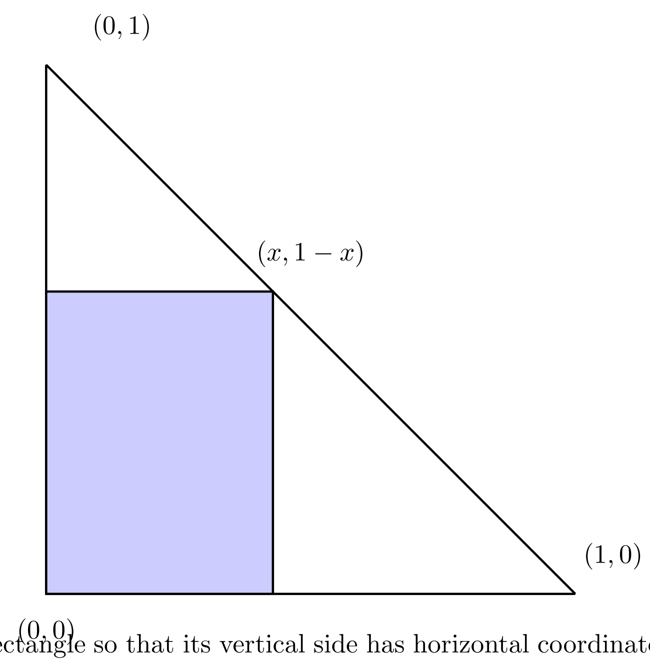
4. A right isosceles triangle is to have two rectangles inscribed in it with sides parallel to the legs of the triangle. This situation is shown in the picture.



How may this be done to make the combined areas of the regtangles as large as possible?

Solution: Let us first assume that the triangle has legs of length 1 and introduce coordinates as indicated in the illustration.

Before formulating the full problem, we will try to understand a simpler version. What is the best way to inscribe a single rectangle?

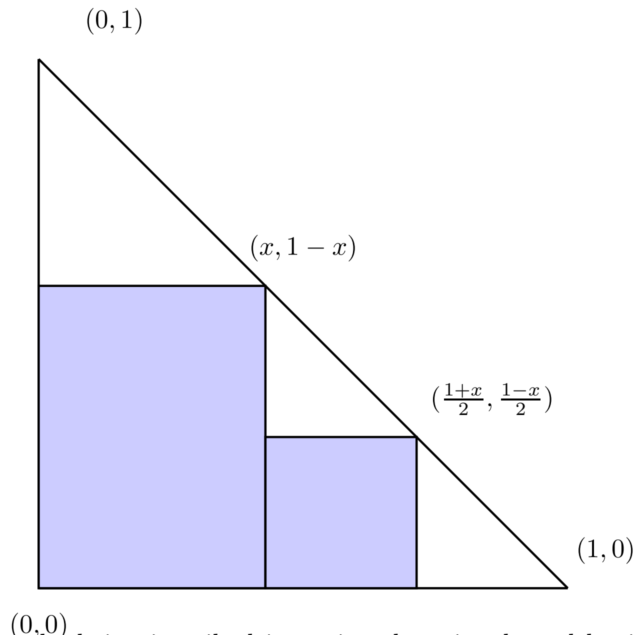


If we construct the rectangle so that its vertical side has horizontal coordinate x, then the height of the rectangle is $1-x$, the area is $A(x) = x(1-x)$.

This is the equation of a parabola that "opens downward" and with roots at 0 and 1. So the maximum value of A is obtained when x lies halfway between the roots, namely at $x = 1/2$.

In summary, to inscribe a rectangle of maximal area in a right isosceles, make the sides of the rectangle equal to 1/2 the length of the triangle's side. Then the area will be 1/2 the area of the triangle.

Now suppose that we've solved the more general problem and found that the right edge of the left rectangle must be at horizontal coordinate x as shown in the picture:



Then the second rectangle, being inscribed in an iscoeles triangle and having as large an area as possible, must have its horizontal side with length 1/2 then length of its triangle — that is, with horizontal coordinate $(1/2)(1+x)$. and its area will be $(1/4)(1-x)^2$. This follows from our previous conclusion.

Therefore the maximum possible area of the two rectangles is

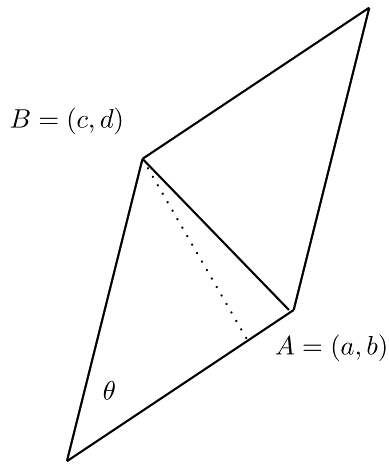
$$K(x) = x(1-x) + ((1-x)/2)^2 = (-1/4)(3x^2 - 2x - 1).$$

This is again a quadratic that "opens down." It has roots at $x = 1$ and $x = -1/3$ and therefore its maximum occurs at the point midway between them, namely $x = 1/3$.

The maximum area is then obtained by using regangles with bases $[0, 1/3]$ and $[1/3, 2/3]$ on the x-axis. The areas of these rectangles sum to 1/3.

5. The parallelogram P has vertices at the points $(0, 0)$, (a, b), (c, d) and $(a + c, b + d)$. Show that the area of P is $|ad - bc|$.

Solution:



The principle difficulty is to determine $\sin(\theta)$, and we can do this starting with the law of cosines:

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos(\theta).$$

We write this in coordinates, using the Pythagorean Theorem for the formula giving the distance between two points, as

$$(a-c)^2 + (b-d)^2 = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos(\theta),$$

so that

$$\begin{aligned}\cos(\theta) &= \frac{a^2+b^2+c^2+d^2-(a-c)^2-(b-d)^2}{2\sqrt{a^2+b^2}\sqrt{c^2+d^2}} \\ &= \frac{2bd+2ac}{2\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.\end{aligned}$$

Then

$$\begin{aligned}\sin(\theta) &= \sqrt{1-\cos^2(\theta)} \\ &= \sqrt{1-\left(\frac{2bd+2ac}{2\sqrt{a^2+b^2}\sqrt{c^2+d^2}}\right)^2} \\ &= \sqrt{\frac{4(a^2+b^2)(c^2+d^2)-(4)(bd+ac)^2}{4|OA|^2|OB|^2}} \\ &= \sqrt{\frac{(ad-bc)^2}{|OA|^2|OB|^2}} \\ &= \frac{|ad-bc|}{|OA||OB|}.\end{aligned}$$

Then the area of P is $|OA||OB|\sin(\theta) = |ad-bc|$.