Write your answers in the Blue Book. Print your names & write the number of students taking this test in the upper right corner of the Blue Book. Put this test & the Blue Book in the provided envelope.

1. (10 points) What is the ones digit in 3^{2007}?

Solution: There is a pattern,

$$\begin{array}{llll} 3^0 = 1 & 3^1 = 3 & 3^2 = 9 & 3^3 = 27 \\ 3^4 = 81 & 3^5 = *3 & 3^6 = *9 & 3^7 = *7 \\ \cdots & & & \\ 3^{2004} = 1 & 3^{2005} = *3 & 3^{2006} = 9 & \mathbf{3^{2007}} = *7 \end{array}$$

2. (10 points) Mary randomly selects one number from among the numbers

$$1, 2, 3, 4, \cdots, 98, 99, 100.$$

What is the probability that the number she selects is divisible by both 2 and 3. Express your anwer as a fraction.

Solution: A number divisible by 2 and 3 is divisible by 6 and vice versa. There are 16 multiples of 6 between 0 and 100. So the probability is 16/100=4/25.

3. (10 points) The circle in figure 1 has a diameter of 20 cm and the triangles in the figure are equilateral.
   (a) Find the area of the region inside the circle, but outside the hexagon.
   (b) What percent of the total area of the disc is this?

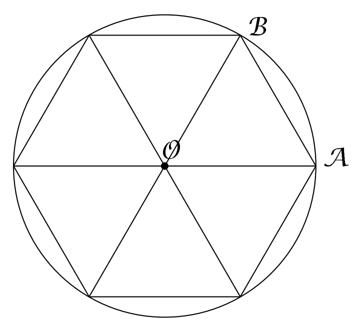

Figure 1: For question # 3.

Solution: $\angle\mathcal{AOB}$ is 60 deg. Since $10 = \overline{\mathcal{OA}} = \overline{\mathcal{OB}}$ because they are radii, we know that triangle $\mathcal{OAB}$ has at least 2 equal sides, so the base angles are equal. Thus all of the angles in triangle $\mathcal{OAB}$ are equal and hence all the sides are equal. Next drop a perpendicular to the base $\mathcal{AB}$, it bisects the base and an easy use of Pythatgaoras' theorem yields the altitude of triangle $\mathcal{OAB}$ is $5\sqrt{3}$. So the area of this triangle is $(1/2) \cdot 5\sqrt{3} \cdot 10 = 25\sqrt{3}$. Thus the hexagon has area $6 \times 25\sqrt{3} = 150\sqrt{3}$. The area of a circle of radius 10 is 100π. The answer to (a) is $100\pi - 150\sqrt{3} \approx 54.35$. (b) The ratio of the areas is$(100\pi - 150\sqrt{3})/(100\pi) \approx .1730 = 17.30\%$.

4. (10 points) Will gives Peter and Maria as many peanuts as each already has. Then Peter gives Will and Maria as many peanuts as each of them already has. Finally, Maria, gives Will and Peter as many peanuts as each has. If at the end each has sixteen peanuts, how many peanuts did each have at the beginning?

Solution: Work backwards, start with what you know.

Round	Will	Peter	Maria
Final	16	16	16
Maria	8	8	32
Peter	4	28	16
Will	26	14	8

Answer $\boxed{\text{Will has 26 peanuts, Peter has 14 peanuts and Maria has 8 peanuts.}}$

5. (10 points) Triangle $\mathcal{ABC}$ in figure 2, has the property that the length of side $\mathcal{AB}$ is the length of side $\mathcal{BC}$. Assume $\angle\mathcal{ADC}$ is a right angle. Prove $2 \times \angle\mathcal{CAD} = \angle\mathcal{ABC}$.

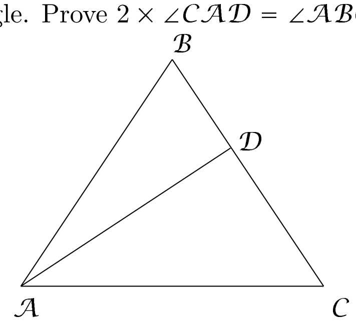

Figure 2: For question # 5.

Hint: Let $\alpha = \angle\mathcal{CAD}$, $\alpha' = \angle\mathcal{DAB}$, $\beta = \angle\mathcal{ABC}$ and $\gamma = \angle\mathcal{ACB}$. Explain why

$$\alpha + \alpha' = \gamma \tag{1}$$
$$\beta + \alpha' = 90 \text{ deg} \tag{2}$$
$$\gamma + \alpha = 90 \text{ deg} . \tag{3}$$

For partial credit, you may assume these facts and prove the result.

Solution: Triangle ABC is isosceles, so $\angle BAC = \angle BCA$. This is equation (1). Equation (2) follows because there are 180 degrees in a triangle such as triangle ABD, but $\angle ADB = 90$ deg. Thus the other 2 angles sum to 90 deg. Apply this argument to triangle ACD and this establishes equation (3).

We are trying to prove $2\alpha = \beta$. One way to do this, is to add eq. (1) to (3), so we have

$$2\alpha + \gamma + \alpha' = \gamma + 90$$

Cancel the γ

$$2\alpha + \alpha' = 90$$

Now use eq. (2)

$$2\alpha + \alpha' = \beta + \alpha'$$

Finally, cancel the α'

$$2\alpha = \beta.$$