Write your answers in the Blue Book. Print your names & write the number of students taking this test in the upper right corner of the Blue Book. Put this test & the Blue Book in the provided envelope.

1. (10 points) Consider triangle $\mathcal{ABC}$ with $\mathcal{TRQP}$ an inscribed square. Find the area of the inscribed square as a function of the base and the height of triangle $\mathcal{ABC}$.

   Follow these steps, if you can not do one of the steps you are still able to use it to a further step, -

   (a) Why is triangle $\mathcal{TRC}$ similar to triangle $\mathcal{ABC}$?

   (b)
   - Let $h = \overline{\mathcal{CD}}$ be the altitude of triangle $\mathcal{ABC}$.
   - Let $h' = \overline{\mathcal{CS}}$ be the altitude of triangle $\mathcal{TRC}$.
   - Let $b = \overline{\mathcal{AB}}$ be the base of triangle $\mathcal{ABC}$.
   - Let $b' = \overline{\mathcal{TR}}$ be the base of triangle $\mathcal{TRC}$.

   Explain why the next two equations hold:

$$\frac{h'}{h} = \frac{b'}{b}$$
$$h' = h - b'$$

   (c) Solve for b' in terms of b and h.

   (d) Compute the area of $\mathcal{TRQP}$ as a function of b, h.

Solution:

(a) The two triangles have the same angles, since they share an angle and their bases are parallel.

(b) The first equation is uses the theorem that in similar triangles the corresponding sides of these triangles are proportional. The second equation is clear.

(c) $b'h = bh' = (h - b')b = bh - b'b$. So $b'(b + h) = bh$ or $b' = bh/(b + h)$.

(d) Area($\mathcal{TRQP}$) $= (bh/(b + h))^2$.

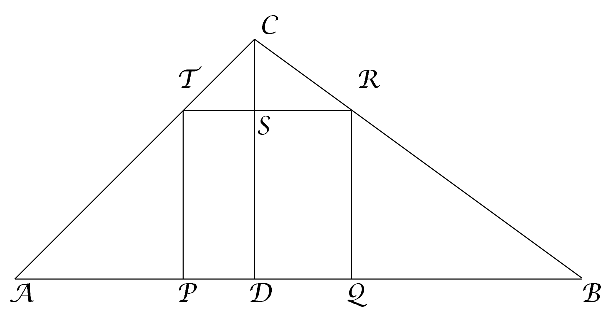

Figure 1: Use this figure for question 1.

2. (10 points) If Mary gets a 97 on the next math test her (arithmetic) average will be 90. If she gets a 73, her average will be 87. How many tests has Mary already taken?

Solution: Let x = the number of tests Mary has taken and let T be the total of her scores on these tests. Then

$$\frac{97+T}{x+1} = 90$$
$$\frac{73+T}{x+1} = 87$$

Thus $(97+T)/(x+1) = (73+T)/(x+1) + 3 = (73+T+3x+3)/(x+1)$. So $x = 7$.

3. (10 points) Consider the parabolas with equations $y = \frac{2}{9}(x+1)^2 + 2$ and $y = x^2 + 4x$. Let $\mathcal{R}$ be the quadrilateral with two vertices at the x-intercepts of the second curve and its other two vertices at the vertices of the parabolas.
   (a) Sketch the two curves on a single coordinate system.
   (b) Find the area of $\mathcal{R}$.
   (c) Is $\mathcal{R}$ a trapezoid or not? Prove your conclusion.

Solution:

(a) See figure 2. The coordinates are $\mathcal{A}(-4,0)$, $\mathcal{B}(-2,-4)$, $C(0,0)$ and $\mathcal{D}(-1,2)$.

(b) Area($\mathcal{R}$)=Area(triangle$\mathcal{ACD}$)+Area(triangle$\mathcal{ABC}$) $= (1/2)\cdot 4\cdot 2 + (1/2)\cdot 4\cdot 4 = 12$.

(c) A trapezoid has parallel sides and sides $\mathcal{AB}$ and $C\mathcal{D}$ have the same slope, which is -2. Hence these sides are parallel.

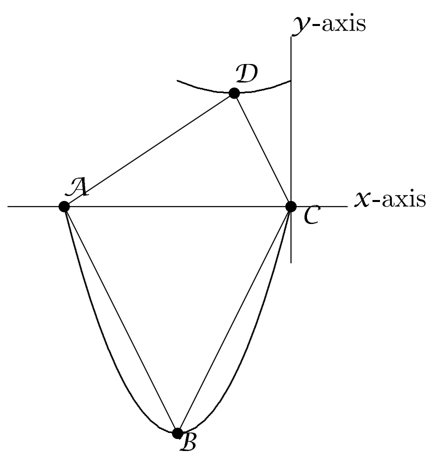

Figure 2: Use this figure for question 3.

4. (10 points) Solve for x

$$\sqrt[6]{8} + \sqrt[3]{x} = \frac{7}{3-\sqrt{2}}.$$

Be sure to simplify your answer as much as possible.

Solution: Note $\sqrt[6]{8} = \sqrt{2}$.

$$\sqrt[3]{x} = \frac{7}{3 - \sqrt{2}} - \sqrt{2} = \frac{9 - 3\sqrt{2}}{3 - \sqrt{2}} = \frac{3\left(3 - \sqrt{2}\right)}{3 - \sqrt{2}} = 3.$$

So

$$x = 3^3 = 27.$$

5. (10 points) A wire is in the form of a circle whose area is $3\sqrt{3}$ square meters. If the wire is rebent into the shape of an equilaterial triangle, find the area of this triangle. Be sure to simplify your answer as much as possible.

Solution: By Pythagoras' theorem the area of an equilateral triangle of side length h is $(\sqrt{3}h^2)/4$. The circle has radius

$$r = \sqrt{\frac{3\sqrt{3}}{\pi}}.$$

Its circumference is

$$2\pi r = 2\pi \cdot \sqrt{\frac{3\sqrt{3}}{\pi}} = 2\sqrt{3\pi\sqrt{3}}.$$

Hence the side length of the triangle is

$$h = (2/3)\sqrt{3\pi\sqrt{3}}.$$

Plug this into the area of the triangle formula above and we get an answer of π.