Write your answers in the Blue Book. Print your names & write the number of students taking this test in the upper right corner of the Blue Book. Put this test & the Blue Book in the provided envelope.

1. (10 points) The mathemagician asks Alice to take out pencil and paper and

   1. secretly write down a three digit number with the digits decreasing, (So 432 is ok, 423 is not ok.)
   2. reverse the order and subtract from the first number,
   3. take the answer and add it to the reverse of itself.

   For example, say Alice picks 432.

   Table 1: Is it magic?

	4	3	2
−	2	3	4
	1	9	8
+	8	9	1
1	0	8	9

   The mathemagician (somehow) knows that 1089 will always be the final answer. Of course, Alice is amazed when she learns that her secret is known. This is not magic, it is better, it's algebra. What is the algebraic explanation? [1]

   Solution: Let $a \cdot 10^2 + b \cdot 10 + c$ be Alice's number, with $a > b > c > 0$. Then

   $$(a \cdot 10^2 + b \cdot 10 + c) - (c \cdot 10^2 + b \cdot 10 + a) = 99(a - c).$$

   Note if $a - c = 1$, then $b = a$ or $b = c$. So $a - c = 2, 3, 4, 5, 6, 7, 8, 9$, hence $99(a - c) = 198, 297, 396, 495, 594, 693, 792, 891$. Then check that $198 + 891 = 297 + 792 = 396 + 693 = 495 + 594 = 1089$.

2. (10 points) There are several ways to compute the *average* or the *mean* of two numbers $a > 0$ and $b > 0$.

   - The *Arithmetic Mean* of a and b is
   $$\mathcal{A}(a, b) = \frac{a + b}{2}.$$
   - The *Geometric Mean* of a and b is
   $$\mathcal{G}(a, b) = \sqrt{ab}.$$

   (a) Find reasons to explain why $\mathcal{G}(a, b) \leq \mathcal{A}(a, b)$.
   Hint: Consider a right triangle with hypotenuse of length $(a + b)/2$ and one leg of length $|a - b|/2$. What is the length of the the other leg?

   (b) What needs to be true about a and b to make $\mathcal{G}(a, b) = \mathcal{A}(a, b)$?

[1] Problem # 1 appears in *Secrets of Mental Math*, by A. Benjamin, M. Shermer(New York:Three Rivers Press:2006), p.199. Two copies of this book will be awarded as door prizes today at 1 PM.

Solution: (a) Let the length of the other leg be x. Then we know

$$x^2 + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$$
$$x^2 + \frac{a^2}{4} - \frac{2ab}{4} + \frac{b^2}{4} = \frac{a^2}{4} + \frac{2ab}{4} + \frac{b^2}{4}$$
$$x^2 = \frac{2ab}{4} + \frac{2ab}{4} = ab$$

So $x = \sqrt{ab}$ and since the hypotenuse is the longest side in a right triangle, we have the result.

(b) For one side of a right triangle to have the same length as the hypotenuse, the other side must have length 0. So $|(a-b)/2| = 0$ or $a = b$.

3. (10 points) Find all real numbers x such that

$$(1-|x|)(1+x) \gneq 0$$

Solution: Case 1. $0 < 1 - |x|$ and $0 < 1 + x$. So $|x| < 1$ and $-1 < x$. This is the open interval from -1 to 1=$(-1, 1)$.

Case 2. $0 > 1 - |x|$ and $0 > 1 + x$. So $|x| > 1$ and $-1 > x$. This is the open interval from $-\infty$ to $-1 = (-\infty, -1)$. Answer:$(-\infty, -1) \cup (-1, 1)$

4. (10 points) Bob and Alice are playing a coin tossing game. Alice tosses first, then they take turns. The rule is:

The first person to toss a head immediately after the other person tossed a tails is the winner.

(a) What is the probability that Alice wins on her first toss?
(b) What is the probability that Bob wins on his first toss?
(c) What is the probability that Alice wins on her second toss?
(d) What is the probability that Bob wins on his fifth toss?
(e) What is the probability that Bob wins on his k-th toss?

Solution:

(a) 0, as her first toss is not preceded by a T.

(b) 1/4 since this happens only if the first two throws result in TH.

(c) Prob(HTH)+Prob(TTH)=1/8+1/8=1/4.

(d) Prob(Bob wins on his 5th toss)=9/1024.

(e) Prob(Bob wins on the k-th toss) is

$$\sum_{j=0}^{2k-2} \text{Prob}(\underbrace{HH..H}_{j\text{ H's}} \overbrace{T...T}^{2k-j-1\text{ T's}} H),$$

where the sequence starts with j consecutive heads followed by $2k - j - 1$ consecutive tails, followed by a single H. So the answer is the number of terms in the sum times 2^{-2k} which is

$$(2k - 1)/4^k.$$

Note if $k = 5$ then $(2 \cdot 5 - 1)/4^5 = 9/1024$.

5. (10 points) In the figure below, the radii $\overline{\mathcal{AD}}$ and $\overline{\mathcal{BC}}$ are equal to ρ. We further assume $\overline{\mathcal{DC}}$ is a common tangent to both circles and that the area of region w equals the area of region v. Find the area of the quadrilateral $\mathcal{ABCD}$.

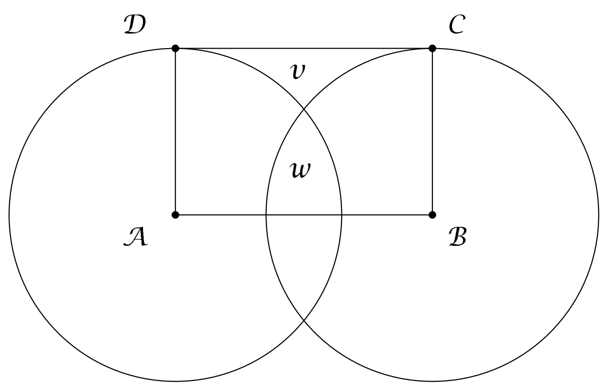

Figure 1: For question 5.

Solution: The area of quadrilateral $ABCD$ = area of region u+area of region w+area of region z + area of region v= (area of region u+area of region w)+(area of region z + area of region w)=$\pi \cdot \rho^2/4 + \pi \cdot \rho^2/4 = \pi \cdot \rho^2/2$.