PART II

Instructions: Write your solutions in the blue book. Remember that you must explain your answers. Even correct answers without complete explanations and justifications may receive no credit! And even if you can't solve a problem completely, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

6. If f is continuous and $\int_0^9 f(x)\,dx = 4$ find the value of $\int_0^3 xf(x^2)\,dx$.

Solution: Try integrating $\int xf(x^2)\,dx$ by substitution with $u = x^2$ so $\frac{du}{dx} = 2x$. Then

$$\int xf(x^2)\,dx = \int f(u)\frac{1}{2}\frac{du}{dx}\,dx = \frac{1}{2}\int f(u)\,du.$$

If $F(x)$ is some anti derivative of f, so that $\int_0^9 f(x)\,dx = 4 = F(9) - f(0)$ then

$$\begin{aligned}\frac{1}{2}\int_0^3 f(u)\,du &= \frac{1}{2}F(u)|_{x=0}^{x=3}\\ &= \frac{1}{2}F(x^2)|_{x=0}^{x=3}\\ &= \frac{1}{2}(F(9) - F(0))\\ &= 2.\end{aligned}$$

7. Eight people want to ride to Lake Erie to go birdwatching. Environmentally concerned, they intend to carpool and ride up in two cars. One car is a van that seats 5 people. The other car is a sedan that seats 4 people. In how many different ways can these birders ride to the lake if the only thing that counts is "who rides with whom"?

Solution: Introduce any other person, anonymous, invisible, but seat occupying, into the gruops of 8 to make 9 distinct riders. Pick any five of these to ride in the van and assign the the others to the car. This can be done in $\binom{9}{5}$ ways.

8. Solve the following equations for x:

   (a) $\sin(2\arctan(x)) = 1$

Solution: If $\sin(2\arctan(x)) = 1$ then $2\arctan(x) = \pi/2 + 2k\pi$ for some integer k, and so $\arctan(x) = \pi/4 + k\pi$. Values of x that satisfy this equation are

$$x = \tan(\pi/4 + k\pi),\ k \text{ and integer.}$$

Of course, this simply says $x = 1$.

(b) $\log_{10}(10\log_{10}(\log_{10}(x^{-10}))) = 1$.

Solution: Calculate:

$$\begin{aligned}
\log_{10}(10\log_{10}(\log_{10}(x^{-10}))) &= 1 \\
10\log_{10}(\log_{10}(x^{-10})) &= 10 \\
\log_{10}(\log_{10}(x^{-10})) &= 1 \\
\log_{10}(x^{-10}) &= 10 \\
-10\log_{10}(x) &= 10 \\
\log_{10}(x) &= -1 \\
x &= 10^{-1} \\
x &= \frac{1}{10}
\end{aligned}$$

9. A biologist wants to study the extent to which birds "cluster together" when they build their nests. She plans to do an experiment by visiting 10 backyards each spring and recording the number of bird nests in each yard. Thus, the results of her observations will be a collection of numbers $\{x_1, \ldots, x_{10}\}$ recording the number of nests found in each yard. She needs a formula to compute how "clustered" the nests are. If all the nests are in one yard (so that all of the x_i's are 0 except one of them), she wants the formula to produce a very large number. If all the x_i's are equal, the formula should produce a smaller number, since in this case, the nests aren't so clustered. Your problem is to propose a formula with which the biologist can measure "clustered-ness". Your answer should include a description of the properties of your formula which make it a good measure of "clustered-ness".

Solution: There are a variety of ways to answer this question. One possible formula to use is the variance or standard deviation of the x_i. For this it is pretty easy to show that the formula is smallest when the x_i are equal (no clustering) and positive otherwise.

Another solution offered was $\max x_i - \min x_i$ where, again, we get the value 0 if the x_i are all equal and a positive number if they are not all equal.

10. Here's a description of a "magic" trick you can use to amaze your friends. Tell them to start with any positive integer and count the number E of even and the number O of odd digits. Have then write these numbers down next to each other with O to the right of E and followed by their sum $E + O$.

For example, starting with 43435 they'd have $E = 2$, $O = 3$, $O + E = 5$ and the new number would be 235.

Have them treat this result as a new number and repeat the process until they arrive at a number that isn't changed by the procedure.

You can then amaze your friends them by telling them what number they've arrived at. This is possible because it is always the same number, no matter what they started with.

(a) What is this number?

Solution: 123. You can find this number by applying the procdure to 235 to obtain 123 and then observing that, application to 123 leaves it unchanged.

(b) Can you explain why the procedure must always reach this number eventually?

Solution: A complete justification of this assertion naturally proceeds in a number of steps. First, show that whatever number one starts with one eventually reaches a 3-digit number. Then argue that from every 3-digit number one eventually reaches 123/

For convenience, let's denote by $f(n)$ the number obtained by applying the procedure to n.

First consider the case in which n has 3 digits so we may write $n = n_3 n_2 n_1$ (with each n_i a single digit). Then $f(n)$ must end in a 3. The only possible values for $f(n)$ are

$$303, 213, 123, \text{ and } 033,$$

and applying f to each of these produces 123.