UC Math Competition ADVANCED PROBLEMS March 2007

PART I

Instructions: Write your solutions in the blue book. Remember that you must explain your answers. Even correct answers without complete explanations and justifications may receive no credit! And even if you can't solve a problem completely, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

1. Can you find a positive integer so that one fifth of it is a perfect square and one half of it is a perfect 5th power?

Solution: Suppose that m were such a number. Then there are integers a and b so

$$\begin{aligned} m &= 2a^5 \\ m &= 5b^2 \end{aligned}$$

If this is the case, then $2a^5 = 5b^2$. Should primes other than 2 and 5 be factors of m, and thus of a and b, they must occur on both sides of this equation with equal powers and so may be divided out. Therefore we may as well proceed under the suppostition that the only divisors of a and b are 2 and 5. That is

$$a = 2^{a_2}5^{a_5} \text{ and } b = 2^{b_2}5^{b_5}.$$

The equation $2a^5 = 5b^2$ then reads

$$22^{5a_2}5^{5a_5} = 52^{2b_2}5^{2b_5},$$

or,

$$2^{5a_2+1}5^{5a_5} = 2^{2b_2}5^{2b_5+1}.$$

We must have

$$\begin{aligned} 5a_2 + 1 &= 2b_2 \\ 5a_5 = 2b_5 + 1. \end{aligned}$$

These equations can be satisfied, for example, by taking

$a_2 = 1$, $b_2 = 3$, $b_5 = 2$ and $a_5 = 1$.

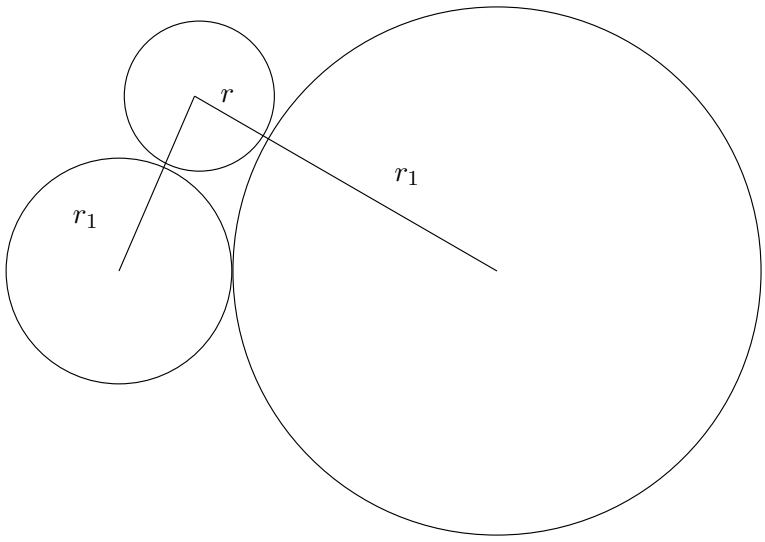
Then $a = 2^{a_2}5^{a_5} = 2^15^1$, and, of course, $b = 2^{b_2}5^{b_5} = 2^35^2$.

This tells us that

$$m = 2a^5 = 2^65^5 = 5b^2.$$

2. Let C_1 and C_2 be two circles which are tangent at the point p, and assume that neither circle is contained in the other. Let S be the set of points (x, y) with the property that there is a circle centered at (x, y) which is tangent to both C_1 and C_2 at points other than p. Show that S is either part of a line or part of a hyperbola.

Solution: When two circles are tangent at a point the radii of the circles connecting their centers to this point are colinear as both have the point in common and both are orthogonal to the tangents to the circles at the point.

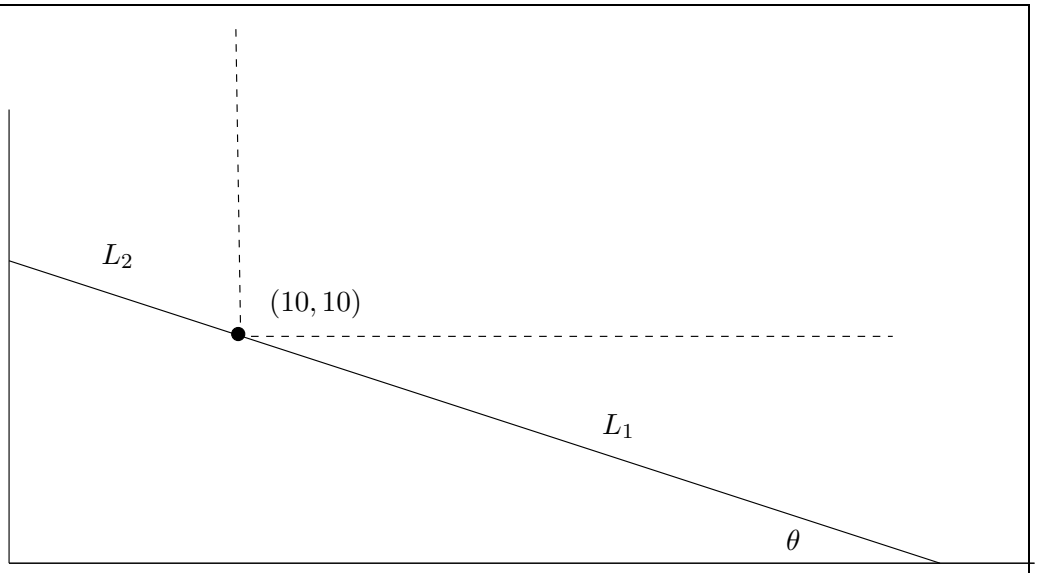


Let A_1 and A_2 be the centers of the circles and X the center of the circle with coordinates (x, y). Then $|X - A_1| = r + r_1$ and $|X - A_2| = r - r_2$, so the difference in the distance from X to A_1 and X to A_2 is $r_1 + r - (r_2 - r) = r_1 - r_2$.

Unless it is a perpendicular bisector (a line) the locus of points with the property that the difference of the distance from two fixed points is constant is a hyperbola.

3. A rigid beam of length L (and having width and height that can be ignored) must be held perfectly horizontal as it is carried around a corner where two corridors meet. If the corridors are 10 feet wide, what is the largest possible value of L for which this can be accomplished?

Solution:



An interesting point about this questionis that it appears to ask for a maximum (the longest beam) but in reality we need to find a minimum — the shortest hypotenuse of a right triangle that has legs on the coordinate axes, right angle at the orgin and has hypotenuse passing through the point $(10, 10)$. These triangles are most conveniently parametrized by the angle θ beween the hypotenuse and one of the legs, as shown in the figure.

Then $L_1 = 10\csc(\theta)$ and $L_2 = 10\sec(\theta)$ so the length we want to minimize is

$$L(\theta) = 10(\csc(\theta) + \sec(\theta)).$$

For this function

$$L' = \frac{\sin^3(\theta) - \cos^3(\theta)}{\sin^2(\theta)\cos^2(\theta)},$$

and this derivative is 0 when $\theta = \pi/45$. The extreme is an isosceles triangle with legs of length 20 and hypotenuse of length $20\sqrt{2}$.

4. In the game "Spill and Spell" a player rolls 10 dice. Each face of each die has a letter on it. The player tries to arrange the dice to form words which are connected in a cross-word fashion. Play stops after 3 minutes, and the player's score is computed according to the formula

$$S = (x_1)^2 + (x_2)^2 + \ldots, +(x_n)^2 - x_R$$

where $x_1, \ldots, x_n$ are the number of letters in the words formed by the player and x_R is the number of dice the player was unable to incorporate into a word. What are the smallest and largest scores that a player can attain in this game?

Solution:

We're told that $x_1 + \cdots + x_n + x_R = 10$ where n is the number of words formed. Clearly the maximum value of $S = (x_1)^2 + (x_2)^2 + \ldots, +(x_n)^2 - x_R$ occurs when $x_R =$ and the minumum when $x_R = 10$.

The interesting part of the question is how the letters should be distributed among the words to make the sum of the squares of their lengths as large as possible.

Consider the case where $n = 2$ to begin with. If $x_1 + x_2 = 10$ what should x_1 and x_2 be to make the sum of their sqaures large? Well, $x_2 = 10 - x_1$ and so the thing we want to maximize is

$$x_1^2 + (10 - x_1)^2 = 2x_1^2 - 20x_1 + 100.$$

The minumum of this parabola is at $x_1 = 5$ and the maximum (on the interval of interest) occurs at the endpoints. To get the highest score we should make a single word out of all 10 letters rather than dividing them into groups and making two words.

This same principle applies to paritioning any number into two and maximizing the sum of the squares of the parts. This obersvation tells us that we should use all 10 letters to make a single word is we want a maximum score. The highest possible score is 100 nd the smallest is -10.

5. It is easy to write an equation describing a curve which divides the plane into two pieces. For example, the equation $x^2 + y^2 = 1$ has a graph which is a circle: it has an inside and an outside — two pieces. Write an equation the graph of which divides the plane into *four* pieces.

Solution: How about the equation $xy = 0$? This has, as its graph, the two coordinate axes. These lines divide the plane into four quadrants.

Or perhaps the equation $|x| = |y|$, whose graph also consists of a pair of perpendicular lines? In polar coordinates the curve $r = \sin(3\theta)$ is a 3-leaved figure and so it, too, cuts the plane into four regions.