Middle Level: **Part 2** The 21$^{\text{th}}$ UC Math Bowl March 4, 2006

Write your answers in the Blue Book. Print your names & write the number of students taking this test in the upper right corner of the Blue Book. Put this test & the Blue Book in the provided envelope.

1. (10 points) A man can do a job in 9 days and his son can do the same job in 16 days. They start working together. After 4 days the son leaves and the father finishes the job. How many more days does it take the father to finish the job?

Solution: The man does 1/9-th of the job/day & the son does 1/16-th of the job/day. Together they do (1/9+1/16)=25/144-th of the job/day. So after 4 days they have completed $4 \times 25/144 = 25/36$-th of the job. Thus the father has to complete $1 - 25/36 = 11/36$ of the job. We let x be the number of days the father works after his son has left. The equation is

$$\frac{x}{9} = \frac{11}{36}.$$

So $x = 9 \times \frac{11}{36} = \frac{11}{4} = 3.75$ days.

2. (10 points) We have 4 bags of coins. In each bag, all the coins weigh the same (but the coins from different bags are not necessarily the same weight). The coins in any given bag weigh 1,2, or 3 ounces. We take one coin from bag 1, 3 coins from bag 2, 9 coins from bag 3, and 27 coins from bag 4. These 40 coins altogether weigh 95 ounces. Determine the weight of a coin from each bag.

Solution: Let

$x_1 =$ the weight of A coin in bag 1
$x_2 =$ the weight of A coin in bag 2
$x_3 =$ the weight of A coin in bag 3
$x_4 =$ the weight of A coin in bag 4.

So

$$\begin{aligned} x_1 + 3x_2 + 9x_3 + 27x_4 &= 95 \\ 3(x_2 + 3x_3 + 9x_4) &= 95 - x_1 \end{aligned}$$

Note 3 (evenly) divides the left hand side, so it must evenly divide the right hand side. Thus $x_1 = 2$. Now we have

$$\begin{aligned} x_2 + 3x_3 + 9x_4 &= 31 \\ 3(x_3 + 3x_4) &= 31 - x_2 \end{aligned}$$

Note 3 (evenly) divides the left hand side, so it must evenly divide the right hand side. Thus $x_2 = 1$. Next we find x_3

$$\begin{aligned} x_3 + 3x_4 &= 10 \\ 3x_4 &= 10 - x_3 \end{aligned}$$

Note 3 (evenly) divides the left hand side, so it must evenly divide the right hand side. Thus $x_3 = 1$. Finally, $3x_4 = 9$, so $x_4 = 3$.

3. (10 points) The height of a certain flagpole is unknown, but the flag rope that goes to the top of the pole is 2 feet longer than the flagpole. Also this rope just reaches the ground when carried 18 feet from the base of the flagpole. What is the height of the flagpole?

Solution: Let x be the height of the flag pole in feet. The flag rope is $x + 2$ feet long and we know $18^2 + x^2 = (x + 2)^2$. Solving for x reveals that $x = 80$. So the pole is 80 feet tall.

4. (10 points) We say y varies as x, if y/x is constant. Prove that if y varies as x, then $x^2 - y^2$ varies as xy.

Solution: So there is a constant C such that $y = Cx$. So $x^2 - y^2 = (1/C) \cdot (y) \cdot x - y \cdot (Cx) = (1/C - 1)(xy)$.

Figure 1: Pseudo-Sudoku

	7		6					
				1		9		
			5					
6		4				2	9	8
	8						4	
			9		6			
	3		2		5			

5. (10 points) (Pseudo-Sudoku) The diagram above does not qualify as a Sudoku puzzle, because it can be completed in several different ways. But there is enough information to determine 3 of the grid squares. Remember the rules of Sudoku, each of the nine 3×3 super squares, each row and each column must contain the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$.

   We will refer to the individual squares using two numbers. The first number tells the row (rows go across) and the second tells the column (columns go up and down). So there is a 6 in $(1, 4), (5, 1)$ & $(7, 6)$.

   (a) Explain why only a 1 or a 5 can be in $(5, 2)$.

   (b) If a 5 is put in square $(5, 2)$ what numbers must be put in squares $(5, 4), (5, 5)$ & $(5, 6)$? Can you determine exactly which number goes in each of these squares?

   (c) You are still putting a 5 in $(5, 2)$. Why can the fourth column not be filled in according to the rules of Sudoku?

   (d) You should now be able to identify 2 of the squares and the numbers that must be put in them. What is the row and column addresses of these squares and what values must be put in them?

   (e) There is a third square. What is its row and column address and what value must be put in it?

Solution:

(a) Look at the numbers in the fifth row, second column and left middle supercell. Only 1 & 5 are missing.

(b) If a 5 is put in $(5, 2)$ then 1,3 & 7 are needed to complete the row. You can not determine which number goes in each cell.

(c) To complete the fourth column we need 1, 3, 4, 7, 8. Now one of the numbers 1, 3 or 7 will be in $(5, 4)$, but then the other 2 of these must be put in $(2, 4)$ and $(8, 4)$. So to finish column 4, 4 & 8 must be put in the 2 open cells $(4, 4)$ and $(6, 4)$. But neither 4 nor 8 can be put in $(6, 4)$, because a number can occur just once in each row, column and supercell.

(d) By (c) we know that a 1 must be put in $(5, 2)$ and 5 must be put in $(5, 5)$.

(e) 1 must be put in $(6, 4)$, because of reasons very similar to the answer for (c).