Write your answers in the Blue Book. Print your names & write the number of students taking this test in the upper right corner of the Blue Book. Put this test & the Blue Book in the provided envelope.

1. (10 points) A volume of a pyramid with a square base of side length b, & height H is

$$\frac{1}{3} \cdot b^2 \cdot H.$$

   (a) The great pyramid of Cheops has $b = 756$ ft & $H = 486$ ft. At a certain point in the building of the great pyramid it was a frustrum of height 283 ft. Find the volume of the frustrum and find the ratio of the volume of the frustrum & the volume of the completed pyramid.

   (b) Find a formula for the volume of the frustrum, as a function of x.

Below is the frustrum of a pyramid, it has a square base of side length b, a top which is also a square of side length a and it is h units high. Also $H = h + x$.

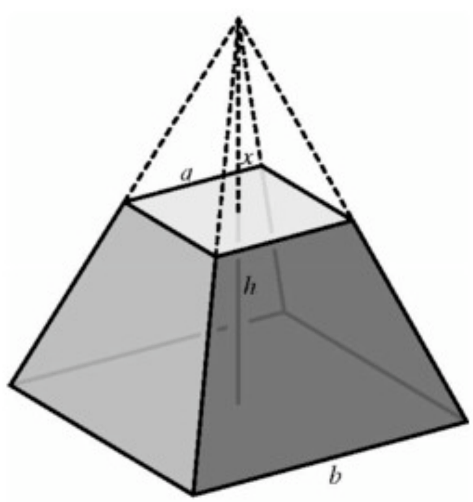

Figure 1: The frustrum of a pyramid.

Solution:

(a) To find the volume of the frustrum just subtract the volume of the pyramid on top from the volume of the whole pyramid. This is $(1/3) \cdot (756)^2 \cdot 486 - (1/3) \cdot (378)^2 \cdot 283 = 79,110,108$.

(b) The ratio is a little smaller than $\frac{7}{8}$ and is about .85.

(c) If we let a be the side length of the top of the frustrum, then $a/x = b/H$. So the volume of the frustrum is

$$\frac{1}{3}\cdot b^2\cdot H - \frac{1}{3}\cdot x^2\left(\frac{b}{H}\right)^2\cdot x = \frac{b^2}{3}\left(H - \frac{x^3}{H^2}\right).$$

2. (10 points) A function$f(x)$ is call *odd* if $f(-x) = -f(x)$. Let

$$f(x) = \ln(\sqrt{x^2+1}+x).$$

(a) Show $f(x)$ is odd.

(b) Show $f(x)$ is $1-1$.

(c) Find the inverse function of $f(x)$.

Solution:

(a)

$$\begin{aligned} f(-x) &= \ln(\sqrt{(-x)^2+1}-x) = \ln\left[\sqrt{x^2+1} - x\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right] \\ &= \ln\left[\frac{1+x^2-x^2}{\sqrt{x^2+1}+x}\right] = \ln\left[\frac{1}{\sqrt{x^2+1}+x}\right] = -\ln[\sqrt{x^2+1}+x]. \end{aligned}$$

(b) We assume $f(a) = f(b)$ & we must show $a = b$. Certainly $a \le b$ or $b \le a$. Assume $a \le b$ so $\sqrt{a^2+1} \le \sqrt{b^2+1}$. Now $\ln[\sqrt{a^2+1}+a] = \ln[\sqrt{b^2+1}+b]$, after apply the exponential function, we have

$$\sqrt{a^2+1}+a = \sqrt{b^2+1}+b.$$

So $\sqrt{a^2+1} - \sqrt{b^2+1} = b - a$, note the left side is ≤ 0 & the right side is ≥ 0. Thus both sides are equal to 0, so $a = b$.

(c) Let $y = \ln(\sqrt{x^2+1}+x)$ now interchange x y and then solve for y. So $x = \ln(\sqrt{y^2+1}+y)$, which means $e^x = \sqrt{y^2+1}+y$. Thus $(e^x - y)^2 = \left(\sqrt{y^2+1}\right)^2$ or $e^{2x} - 2e^x\cdot y + y^2 = y^2+1$. Thus $y = (e^{2x}-1)/(2e^x) = (1/2)(e^x - e^{-x})$. ($y$ is the *hyperbolic sine* and $f(x)$ is *the inverse hyperbolic sine.*)

3. (10 points) Find all of the integers beween 2 & 200, 000, 000, 000, 000 that are perfect squares, perfect cubes and perfect fifth powers, at the same time. For example,

$46,656 = (216)^2 = (36)^3$, so 46,656 is a perfect square and a perfect cube, but it is not a perfect fifth power.

Solution: Let n be such an integer, then n is a product of primes & $2, 3, 5$ have to divide the exponents of each prime. There are just 1 such integer, 2^{30} that is less than 2×10^{14}.

4. (10 points) A line segment of length c moves so that one end point X is on the x-axis & the other, Y, is on the y-axis. Let P be the foot of the perpendicular dropped from the origin, O to the segment.
   (a) Why is $\angle OYP = \angle XOP$?
   (b) Let $\theta = \angle XOP$ & r be the distance between O & P. Explain why
   $$r^2 = c^2 \cos^2\theta \cdot \sin^2\theta.$$
   (c) What famous curve is being traced out by P?

Solution:

(a) Both angles are complements of $\angle POY$.

(b) Since $\theta = \angle POX$, we see $\cos\theta = r/|OX|$. So $|OX|^2 = r^2/\cos^2\theta$. Also $\theta = \angle OYP$ which means $\sin\theta = r/|OY| >$ Thus $|OY|^2 = r^2/\sin^2\theta$. Finally $c^2 = |OX|^2 + |OY|^2 = r^2(1/\cos^2\theta + 1/\sin^2\theta = r^2/(\cos^2\theta \cdot \sin^2\theta$. Thus $r^2 = c^2(\cos\theta \cdot \sin\theta)^2$ & so $r = c\cos\theta \cdot \sin\theta = (c/2)\sin(2\theta)$.

(c) A 4-leaf rose.

5. (10 points) Train F is faster than train S. Also F is 240 meters long and S is 200 m long. If F & S were to pass going in opposite directions in would take 25 seconds, but if they were going in the same direction it would take 3.75 minutes. How fast in kilometers/hour is each train going?

   Hint: Let f be the speed of F in m/sec (meters per second) and s be the speed of S in m/sec. They pass each other going in opposite directions when the backs of their last cars line up and they pass each other going in the same direction when the back of the last car of F lines up with the front of the engine of S.

Solution: Pretend F is going north on the y-axis and S is going south. After 25 sec. F is at $25f$ & its last car is at $25f - 240$ while the engine of S is at $-25s$. So its last car is at $-25s + 200$. So the first equation is $25f - 240 = -25s + 200$. Thus we have $f + s = 88/5$. Next we turn to the equation when they are going in the same direction. If the engine of S is at $225s$ & the last car of F is at $225f - 240$. This yields the second equation $225s = 225f - 240$. So we have the following system of 2 equations in 2 unknowns to solve:

$$f + s = \frac{88}{5}$$
$$f - s = \frac{48}{5}$$

Solving we find $f = 68/5$ m/sec & $s = 4$ m/sec. Thus

$$s = 4\left(\frac{\text{meters}}{\text{second}}\right)\frac{1}{1000}\cdot\left(\frac{\text{kilometers}}{\text{meter}}\right)\cdot\frac{60^2}{1}\left(\frac{\text{seconds}}{\text{hour}}\right).$$

So $s = 14.4$ kilometers/hour & $f = 48.96$ kilometers/hour.