UC Math Competition ADVANCED PROBLEMS March 2006

PART II solutions

Instructions: Write your solutions in the blue book. Remember that you must explain your answers. Even correct answers without complete explanations and justifications may receive no credit! And even if you can't solve a problem completely, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

6. Show that a conic section does not have a point of inflection.

Answer: The appropriate definition of "inflection point" is a point where concavity changes. Recall that it is not necessary to represent a curve as the graph of a function in order to determine its concavity since concavity can be defined in terms of the location of segments of secant lines.

One way to answer this question is to consider, in turn, each of the standard conic sections: ellipse, hyperbola, circle, and parabola. In the neighborhood of each point on these curves, Write the curve as a function, compute the second derivative of the function. Show that the second derivative cannot change sign near the point. To be especially careful, one should check the degenerate conic sections as well, but this isn't too hard. Another approach (it isn't presented here as a complete solution, so we ignore the parabola and degenerate conics) uses implicit differentiation. After a rotation and translation of coordinates, the conic may be written in the form

$$ay^2 + bx^2 = c,$$

where a, b, and c are nonzero. We concentrate on those points of the conic for which $y \neq 0$, so that y is defined implicitly as a function of x. Differentiate implicitly to obtain

$$2ayy' + 2bx = 0$$

and then again:

$$2a(y')^2 + 2ayy'' + 2b = 0.$$

Solving for y'' we obtain

$$y'' = \frac{-2b - 2a(y')^2}{2ay},$$

so that, replacing $\frac{-2bx}{2ay}$ for y', we get

$$y'' = \frac{-8ab(ay^2 + bx^2)}{8a^3y^3} = \frac{-8abc}{8a^3y^3} \neq 0.$$

This shows that on the conic (at least at points where y can be defined as a function of x, the second derivative of y is non-zero. Such points cannot be inflection points. To take care of places where $y = 0$, interchange the role of x and y in the above computation. The result will show that near these points, where x is given locally as a function of y, $x'' \neq 0$.

7. Solve the equation

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x+2.$$

Answer: Using split points at -1, 0, 1 and 2 on the number line, break up the problem into 5 disjoint cases and solve the resulting equations without absolute values. This is really the only way to treat any problem involving absolute values. The solutions are $x = -2$ or $x \geq 2$.

8. How many of each of the ten digits are needed to write out the integers from 1 to $100,000,000$?

Answer: Consider the numbers from 00000000 to 99999999. This list has $100,000,000$ numbers in it, each with 8 digits, so they require $800,000,000$ digits to write down; $80,000,000$ of each digit. To that count, append a 1 and 7 0's for the number $100,000,000$. Finally, remove the $11,111,118$ beginning 0's. The count is then $68,888,889$ 0's, $80,000,001$ 1's and $80,000,000$ of each of the other digits.

9. Find the sum of the infinite series:

$$\sum_{i=1}^{\infty} \frac{i}{10^i} = \frac{1}{10} + \frac{2}{100} + \cdots + \frac{n}{10^n} + \cdots$$

Answer: one way to do tis problem is to notice that the desired sum is $f(1/10)$, where $f(x) = \sum_1^\infty ix^i$. Since we know the formula for the sum of a geometric series with ratio x satisfying $|x| < 1$ is $g(x) = \sum_0^\infty x^i = \frac{1}{1-x}$, and since $g'(x) = \sum_0^\infty ix^{i-1} = \frac{1}{(1-x)^2}$, we see that $f(x) = xg'(x)$, so that $f(1/10) = \frac{1}{10}\frac{1}{(1-(1/10))^2}$. Thus the desired sum is $10/81$.

Another method to evaluate the sum is to write it as:

$$\begin{array}{ccccccccc}
1/10 & + & 1/100 & + & 1/1000 & + & 1/10000 & + & \cdots \\
 & + & 1/100 & + & 1/1000 & + & 1/10000 & + & \cdots \\
 & & & + & 1/1000 & + & 1/10000 & + & \cdots \\
 & & & & & & \vdots & &
\end{array}$$

Each row is a geometric series, and the sums of the rows are $1/9$, $1/90$, $1/900$, and so on. Adding these numbers gives $(1/9)\sum_0^\infty (1/10)^i = 10/81$.