UC Math Competition ADVANCED PROBLEMS March 2006

PART I solutions

Instructions: Write your solutions in the blue book. Remember that you must explain your answers. Even correct answers without complete explanations and justifications may receive no credit! And even if you can't solve a problem completely, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

1. If a slice of pie, in the form of a sector of a circle, is to have perimeter 12 inches, what should be the radius of the circle to make the area of the piece of pie largest?

   Answer: Let c be the perimeter, A the area, r the radius. Then $A = (c/r)r^2)$ and we are given the constraint that $2r + c = 12$. Thus, we seek the maximum of $A(r) = (12 - 2r)r = -2(r^2 - 6r)$. The critical point of the parabola bisects the roots 0 and 6, so the critical point is $r = 3$. The function $A(r)$ is a parabola the leading coefficient of which is negative, so the critical point $r = 3$ is the location of the maximum of the function. The radius of the circle should be 3 in order that the area of the sector be as large as possible.

2. Let P be a parallelogram which has, as three of its vertices, the points $(0, 0)$, (a, b), and (c, d). Express the area of P in terms of a, b, c, and d.

   Answer: The area of the parallelogram is $A = |ad - bc|$. One way to see this is to begin by tilting your head so that the line from $(0, 0)$ to (a, b) becomes horizontal — the x-axis. Then $b = 0$, $|a|$ is the length of the side, and the height of P becomes d or $-d$, which ever is positive. In this case the area is $|ad|$.

   To compute the answer directly, we will find the area of the triangle having the three points as corners, and then double the answer. To find the height of the triangle we seek a point (s, t) on the segment from $(0, 0)$ to (a, b) so that the line connecting (c, d) to (s, t) is orthogonal to the line from $(0, 0)$ to (a, b). That is, we want to solve

$$\begin{aligned} t &= \frac{b}{a}s \\ \frac{(d-t)}{(c-s)} &= -\frac{a}{b} \end{aligned}$$

   for s and t.

   This yields the solution $t = \frac{b}{(a^2+b^2)}(bd + ac)$ and $s = \frac{a}{b}t$. Now compute the distance from (c, d) to (s, t). This is

$$\left((c-s)^2 + (d-t)^2\right)^{1/2} = \left(c^2 + d^2 - 2(d + \frac{ac}{b})t + (1 + \frac{a^2}{b^2})t^2\right)^{1/2}$$

$$= \left(c^2 + d^2 - \frac{(bd+ac)^2}{a^2+b^2}\right)^{1/2}$$

This is the height of the triangle. The base is $(a^2+b^2)^{1/2}$, so twice the area of the triangle is

$$\left(c^2 + d^2 - \frac{(bd+ac)^2}{a^2+b^2}\right)^{1/2} \left(a^2+b^2\right)^{1/2} =$$

$$\left(a^2d^2 + c^2b^2 - 2adbc\right)^{1/2} = \left((ad-bc)^2\right)^{1/2} = |ad-bc|.$$

3. Jill has 45 coins and has 10 pockets in her suit. Can she distribute the coins so that she has **a different number** of coins in each pocket? What if she has 44 coins?

   Answer: If Jill puts 0 coins in one pocket, 1 coin in the next pocket, and 9 coins in her 10th pocket, she will have used $0+1+\cdots+9 = 45$ coins. So she can put 45 coins in her pockets without putting the same number in any two pockets. This also shows that 45 is the smallest number for which this can be done: any sum of 10 distinct, non-negative integers must be at least 45. She cannot achieve the goal with 44 coins.

4. For two positive numbers a and b there are many different notions of "average". This problem is concerned with two averages which we will call A and H. The average A is defined by

   $$A = \left(\frac{a^2+b^2}{2}\right)^{1/2}$$

   and the average H is define by

   $$\frac{1}{H} = \frac{1}{2}(\frac{1}{a} + \frac{1}{b}).$$

   (a) Prove that these averages both lie between a and b.

   (b) Is it true that $H \leq A$ no matter what a and b are? Why or why not?

   Answer: We may assume that $0 < a \leq b$, since otherwise, we simply interchange a and b. Then $a^2 \leq b^2$ so $2a^2 \leq (a^2+b^2) \leq 2b^2$. Dividing by 2 and taking square roots shows that $a \leq A \leq b$. Also, if $0 < a \leq b$, then $1/a \geq 1/b$, so that $2/b \leq (1/a) + 1/b) \leq 2/a$, from which is folows that $a \leq H \leq b$.

It is indeed true that $H \le A$. One way to prove this is to show that $\frac{1}{2}(a+b) \le A$ and $H \le \frac{1}{2}(a+b)$. Proceed as follows:

$$\begin{aligned} 0 &\le (a-b)^2 \\ &= a^2 - 2ab + b^2 \\ &= 2a^2 + 2b^2 - (a+b)^2 \end{aligned}$$

so that $(\frac{1}{2}(a+b) \le \left(\frac{1}{2}(a^2+b^2)\right)^{1/2}$. In the case of $H = \frac{2ab}{a+b}$, we must verify that $\frac{2ab}{a+b} \le \frac{a+b}{2}$. Rearange this inequality, however, and you will obtain $0 \le (a-b)^2$. This proves that, whatever the positive numbers a and b, with $a \le b$, we have $a \le H \le \frac{a+b}{2} \le A \le b$.

5. Prove that for any integer n, the number $n^5 - n$ is a multiple of 5.

   Answer: One method to establish the claim is to use induction. It is easy to verify if $n = 0$ that $n^5 - n$ is evenly divisible by 5. Suppose that $n^5 - n = 5k$ and consider the fact that

$$\begin{aligned} (n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) \\ &= 5(k + (n^4 + 2n^3 + 2n^2 + n)). \end{aligned}$$

   This shows that if $n^5 - n$ is divisible by 5, so is $(n+1)^5 - (n+1)$. The principle of induction lets us conclude that for each non-negative n, the integer $n^5 - n$ is evenly divisible by 5. Since $(-n)^5 - (-n) = -(n^5 - n)$, the fact that $n^5 - n$ is divisible by 5 for $n \ge 0$ implies that it is divisible by 5 for $n \le 0$ too.

   As another method of proof, factor

$$n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n-1)n(n+1)(n^2+1).$$

   In general, a sequence of 5 consecutive numbers can be written as

$$5k,\ 5k+1,\ 5k+2,\ 5k+3,\ 5k+4,$$

   so one of the factors n, $n-1$, or $n+1$ of $n^5 - n$ will be divisible by 5 unless $n = 5k+2$ or $n = 5k+3$ for some integer k.

   If $n = 5k+2$, then $n^2 + 1 = 25k^2 + 20k + 4 + 1 = 5(5k^2 + 4k + 1)$, and if $n = 5k+3$, then $n^2 + 1 = 25k^2 + 30k + 9 + 1 = 5(5k^2 + 6k + 2)$. In either case, the factor $n^2 + 1$ is divisible by 5.