1. It is easy to write an equation describing a curve that divides the plane into two pieces. For example, the equation $x^2 + y^2 = 1$ has a graph which is a circle: it has an inside and an outside — two pieces. Write an equation the graph of which divides the plane into *four* pieces.

   Solution: There are many ways to accomplish this. One method is suggested by the example of the circle. Let us select a polynomial which has exactly three distinct, positive roots. The polynomial $g(x) = (x-1)(x-4)(x-9)$ will do. Now consider the equation $g(x^2 + y^2) = 0$. This equation is satisfied exactly when $x^2 + y^2$ assumes one of the values 1,4, or 9. The graph of this equation is made up of three concentric circles with center at $(0, 0)$ and with radii 1,2, and 3. We can label the four regions into which the graph divides the plane as R_0,R_1,R_2, and R_3 according to how many circles contain the region.

2. Everybody knows that of all the rectangles with fixed area the square has the smallest perimeter. What about other polygons? Of all the triangles with fixed area, which has the smallest perimeter?

   Solution:

   The pleasing answer to this question is that the triangle with smallest perimeter for fixed area is an equilateral triangle. A careful parametrization of the problem lets us justify this assertion using 1 variable calculus.

   Let us put one vertex of the triangle at the origin, a second at $(a, 0)$ on the positive x-axis, and let the third vertex lie at (x, y) with $y > 0$.

   To say that the area of this triangle is A says that $ay = A$. The perimeter of this triangle then depends on x and a:

$$P = a + \sqrt{x^2 + (A/a)^2} + \sqrt{(a-x)^2 + (A/a)^2}.$$

   For each fixed a the value of x that minimizes the perimeter is $x = a/2$. The easiest way to see this is to think about it in terms of Snell's Law and the shortest path from $(0, 0)$ to the horizontal line at height A/a above the x-axis and back to the point $(a, 0)$.

   So the smallest perimeter for a triangle of area A having side of length a is

$$P = a + 2\sqrt{((a/2)^2 + (A/a)^2}.$$

   A one-variable minimization shows that this perimeter is a minimum when $a = \sqrt{2A}/3^{1/4}$.

   The 3rd vertex is then at $(a/2, A/a)$ and the length of one of the other (non-horizontal) sides is

$$\sqrt{\frac{A}{2\sqrt{3}}+\frac{\sqrt{3}\,A}{2}}=\sqrt{2A}/3^{1/4}$$

This equals a, showing that the perimeter is minimized when the triangle is equilateral.

3. A certain number M is twice a perfect cube and 5 times a perfect square. What could M be?

Solution: We're told that there exist a and b so that

$$\begin{aligned} M &= 2a^3 \\ M &= 5b^2 \end{aligned}$$

It is clear from these equations that both 2 and 5 are divisors of M. For simplicity let us assume that a and b are each simply products of powers of 2 and 5 so that

$$\begin{aligned} a &= 2^{a_1}5^{a_2} \\ b &= 2^{b_1}5^{b_2} \end{aligned}$$

In this case we have

$$\begin{aligned} M &= 2^{3a_1+1}5^{3a_2} \\ M &= 2^{2b_1}5^{2b_2+1} \end{aligned}$$

so that, equating the powers of the primes in two expressions for M,

$$2^{3a_1+1}5^{3a_2}=2^{2b_1}5^{2b_2+1},$$

and so weseek non negative integers that satisfy the eqations

$$\begin{aligned} 3a_1+1 &= 2b_1 \\ 3a_2 &= 2b_2+1. \end{aligned}$$

We can find some solutions of these equations easily by inspection. $2b_1-1$ is a multiple of three if $b_1=2$; this would make $a_1=1$. And $2b_2+1$ is a multiple of 3 is $b_2=1$, which would make $a_2=1$. Thus, a possible value of M is

$$M=2a^3=2(2^1 5^1)^3=2000=2\times 10^3=5\times 20^2.$$

4. One fourth the number of minutes from noon 'til now plus half the number of minutes from now 'til noon tomorrow is the time right (number of minutes past noon) now. What time is it?

Solution: Let x be the number of minutes past noon it is right now.

$$x/4 + (12 * 60 + 12 * 60 - x)/2 = x$$

$$12 * 60 - x/4 = x$$

$$12 * 60 * 4/5 = x$$

$$x = 144 * 4 = (120 + 24)4.$$

This is the number of minutes past noon right now. This number of minutes is 8 hours 96 minutes; so the time is 9:36 PM.

5. Find a function whose derivative is $|x|^3$.

Solution: The fundamental theorem assures us, because $|x|^3$ is continous, that

$$\frac{d}{dx}\int_0^x |t|^3\,dt = |x|^3,$$

so that is a fast way to write down a function with the desired property. Of course, we could add any constant to this function and obtain another fuction that also satisfies the condition.

Perhaps you were thinking you needed to find a formula without an integral sign in it? Note that if $x > 0$, the derivative of $x^4/4$ is $x^3 = |x|^3$. On the other hand, if $x < 0$, the derivative of $-x^4/4$ is $-x^3 = |x|^3$, so an exlicit formula for a function with the desired property is

$$f(x) = \begin{cases} x^4/4 & \text{if } x \geq 0 \\ -x^4/4 & \text{if } x < 0 \end{cases}$$