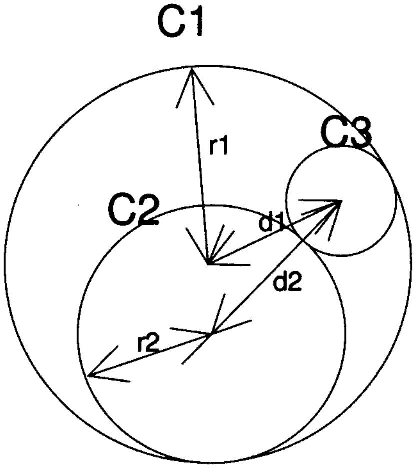
Instructions: Remember that you must explain your answers. Even correct answers without complete explanations and justifications may receive no credit! And even if you can't solve a problem completely, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

1. Suppose that $C_1$ and $C_2$ are two circles arranged as indicated in the firgue: $C_2$ lies inside $C_1$ and is tangent to $C_1$ at a single point.




Let $C_3$ be a circle tangent to both $C_1$ and $C_2$. As shown in the figure, let $r_1$, $r_2$ and $r_3$ be the radii of the three circles and let $d_1$ and $d_2$ be the distance of the center of $C_3$ from the cetners of $C_1$ and $C_2$.

(a) Explain why $r_1 + r_2 = d_1 + d_2$.

**Solution:** First, note that $d_2 = r_2 + r_3$, as can be seen by cutting the line connecting the centers of $C_2$ ans $C_3$ at the point where the circles are tangent. Second, note that $r_1 = d_1 + r_3$, as seen by cutting the radius of $C1$ that passes through the center of $C_3$ at the center of $C_3$.

Then we can just compute $r_1 + r_2$:

$$\begin{aligned} r_1 + r_2 &= (d_1 + r_3) + r_2 \\ &= (d_1 + r_3) + (d_2 - r_3) \\ &= d_1 + d_2 \end{aligned}$$

(b) There are many different circles $C_3$ that can be drawn inside $C_1$, outside $C_2$ tangent to both $C_1$ and $C_2$. Describe the locus of points that are the centers of these circles.

**Solution:** The locus of points $P$ is a collection of points with the property that the sum of the distances from $P$ to the centers of $C_1$ and $C_2$ is the constant $r_1 + r_2$. The collection of all points with this property is an ellipse with foci at the centers of $C_1$ and $C_2$.

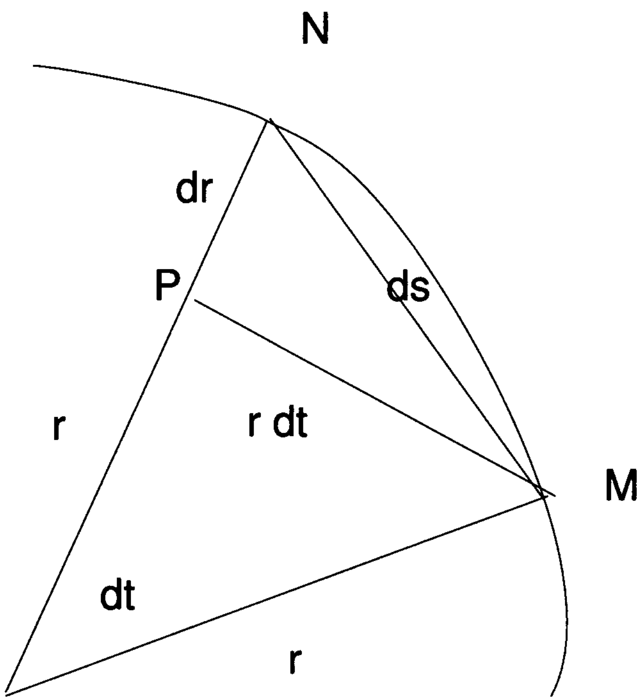
2. In polar coordinates the equation $r = e^\theta$ describes a curve that is known as a logarithmic spiral.

(a) Sketch the graph of this equation.

(b) Find the length of the logarithmic spiral $r = e^\theta$ between the points where $\theta = \theta_0$ and $\theta = \theta_1$

**Solution:** An expression that gives the rate at which arclength changes as $\theta$ changes is

$$\frac{ds}{d\theta} = \sqrt{r^2 + (\frac{dr}{d\theta})^2}.$$




We can derive this relationship (refer to the figure) as follows. Suppose that $M$ and $N$ are two points on the curve $r = f(\theta)$ whose angles differ by the small angle $dt$. Then, of course, the radial distance to $N$ exceeds that of $M$ by about $dr = f'(\theta)d\theta$. Draw $PM$

perpendicular to that radius ON and make the approximation that the distance from $P$ to $M$ is the same as the distance along the arc of a circle of radius $r$ subtended by the angle $dt$. Then we obtain

$$ds^2 = dr^2 + r^2(d\theta)^2,$$

so

$$(\frac{ds}{d\theta})^2 = (\frac{dr}{d\theta})^2 + r^2,$$

showing that

$$\frac{ds}{d\theta} = \sqrt{f(\theta)^2 + (f'(\theta))^2}.$$

Having either remembered or derived this expression, the integral we must evaluate to find the length of the arc is then

$$\int_{\theta_0}^{\theta_1} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

or,

$$\sqrt{2}\int_{\theta_0}^{\theta_1} e^{\theta}\, d\theta = \sqrt{2}(e^{\theta_1} - e^{\theta_0}).$$

(c) Find the length of the part of the curve that spirals in from $(1, 0)$, limiting at the origin.

**Solution:** This is an improper integral. In our previous calculation we take $\theta_1 = 0$ and let $\theta_0$ tend to $-\infty$.

**Solution:**

3. In how many ways can 10 identical pieces of candy be divided among 3 children?

**Solution:**

Imagine writing a string of 12 symbols in a row. 10 of the symbols are x's and 2 are y's. An example is

**xxxyxxxxyxxx**

We can view the y's as dividing the 10 x's into 3 groups of size $x_1$, $x_2$ and $x_3$. For the example string we'd have $x_1 = 3$, $x_2 = 4$ and $x_3 = 3$.

Evidently, each string of x's and y's corresponds to a distribution of candy among the children. Of course, if we distribute the candy any way we like, we can find a string of x's and y's that corresponds to this distribution.

We conclude that the number in question is the same as the number of different string that can be written using 10 x's and 2 y's. Specifying sucha string amounts to giving the locations of the 2 y's in the string, and this simply requires picking 2 locations from the possible 12. So the number of candy distributions is

$$\frac{12!}{2!10!}.$$

4. Prove that two points $A(a, b)$ and $B(c, d)$ are colinear with $(0, 0)$ if and only if $ad - bc = 0$.

   **Solution:** The two points are colinear together with the origin $O(0, 0)$ if and only if the slopes of the segments connecting $O$ to $A$ and $O$ to $B$ are the same. That is, if and only if either these slopes are both undefined, so that both $d$ and $b$ are 0 or

   $$\frac{b}{a} = \frac{d}{c}.$$

   In either of these cases we see that $ad - bc = 0$.

5. Define a sequence of integers $\{a_i : i = 0, 1, 2, \ldots\}$ by $a_0 = 0$ and, for $k \geq 1$, $a_k = 8k - 4$. So $a_0 = 0$, $a_1 = 4$, $a_2 = 12$, and so on.

   Notice that

   $$\begin{aligned} 0 &= a_0 \\ 4 &= a_0 + a_1 \\ 16 &= a_0 + a_1 + a_2 \end{aligned}$$

   Show that this pattern continues by proving that for every $n \geq 0$,

   $$a_0 + a_1 + \cdots + a_n$$

   is a perfect square.

   **Solution:**

   We can do even better bu showing that $a_0 + a_1 + \cdots + a_k = (2k)^2$. One approach is by induction. The table above takes care of the intial case, so by way of induction hypothesis assume that

   $$a_0 + a_1 + \cdots + a_k = (2k)^2$$

   and consider the sum

   $$\begin{aligned} a_0 + a_1 + \cdots + a_k + a_{k+1} &= (2k)^2 + 8(k+1) - 4 \\ &= 4k^2 + 8k + 4 \\ &= 4(k^2 + 2k + 1) \\ &= 2^2(k+1)^2 = (2(k+1))^2 \end{aligned}$$

   As an alternative approach, think of 4 dots arranged in a square of side 2 and area 4. Add 12 dots around the perimeter to produce a square of side

4 and area 16. Next add 20 dots on the perimeter to produce a square of side 6 and area 36. Continue in this way and you will always add $8k - 4$ dots to produce a square with area $4k^2$.