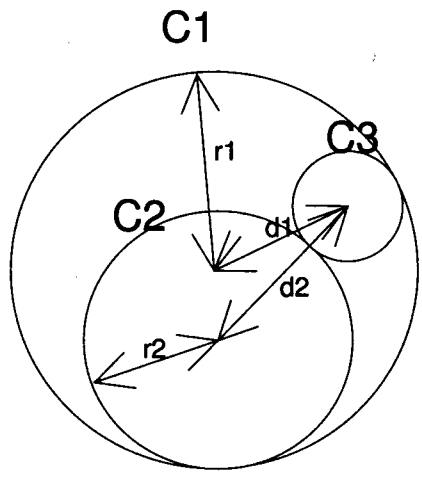
LEVEL I (advanced), part one March 2005

Instructions: Remember that you must explain your answers. Even correct answers without complete explanations and justifications may receive no credit! And even if you can't solve a problem completely, you should carefully explain what you have discovered about the problem since some partial credit may be awarded for your work.

1. Suppose that C_1 and C_2 are two circles arranged as indicated in the figure: C_2 lies inside C_1 and is tangent to C_1 at a single point.



Let C_3 be a circle tangent to both C_1 and C_2. As shown in the figure, let r_1, r_2 and r_3 be the radii of the three circles and let d_1 and d_2 be the distance of the center of C_3 from the centers of C_1 and C_2.

   (a) Explain why $r_1 + r_2 = d_1 + d_2$.

   (b) There are many different circles C_3 that can be drawn inside C_1, outside C_2 tangent to both C_1 and C_2. Describe the locus of points that are the centers of these circles.

2. In polar coordinates the equation $r = e^\theta$ describes a curve that is known as a logarithmic spiral.

   (a) Sketch the graph of this equation.

   (b) Find the length of the logarithmic spiral $r = e^\theta$ between the points where $\theta = \theta_0$ and $\theta = \theta_1$

   (c) Find the length of the part of the curve that spirals in from $(1, 0)$, limiting at the origin.

3. In how many ways can 10 identical pieces of candy be divided among 3 children?

4. Prove that two points $A(a, b)$ and $B(c, d)$ are colinear with $(0, 0)$ if and only if $ad - bc = 0$.

5. Define a sequence of integers $\{a_i : i = 0, 1, 2, \ldots\}$ by $a_0 = 0$ and, for $k \geq 1$, $a_k = 8k - 4$. So $a_0 = 0$, $a_1 = 4$, $a_2 = 12$, and so on.

   Notice that

$$\begin{aligned} 0 &= a_0 \\ 4 &= a_0 + a_1 \\ 16 &= a_0 + a_1 + a_2 \end{aligned}$$

   Show that this pattern continues by proving that for every $n \geq 0$,

$$a_0 + a_1 + \cdots + a_n$$

   is a perfect square.