The 19th Annual UC Math Bowl

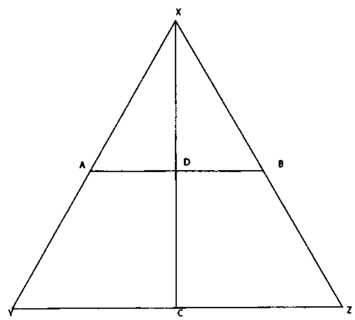
Level 3:Part 2 **Answer Key** March 6, 2004

Answer the questions in the provided Blue Book. Print your names and write the number of students taking this test in the upper right corner of the Blue Book.

1. Georgina loves milk, and she also loves working with numbers. Just for fun she thought of combining a number pattern and the drinking of a gallon of milk. She imagined asking someone to do this: Start by drinking one-half a gallon, then two-thirds of what is left, then three-fourths of what is left, and then four-fifths of what is left, and so on. She thought how cool it would be to figure out exactly how much milk would be left! How much milk did she figure would be left after taking ten drinks?

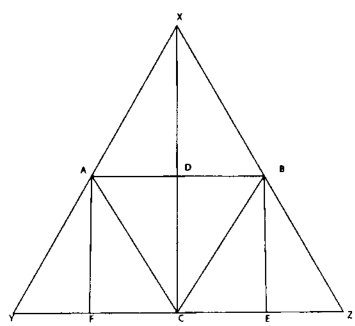
   Answer- It is easier to think about how much is left over. After her 1st drink 1/2 a gal. is left. After the 2nd drink $1/3 \cdot 1/2$ is left. After the 4th drink, $1/4 \cdot 1/3 \cdot 1/2$ is left, and so on, so after 10 drinks there is $1/11 \cdot 1/10 \cdot 1/9 \cdot 1/8 \cdot 1/7 \cdot 1/6 \cdot 1/5 \cdot 1/4 \cdot 1/3 \cdot 1/2 = 1/39{,}916{,}800$ of a gallon left.

2. The area of triangle XYZ is 8 square meters. Points A and B are mid points of congruent segments $\overline{XY}$ and $\overline{XZ}$. Altitude $\overline{XC}$ bisects $\overline{YZ}$. Find the area (in square meters) of the the region $ADCY$.



   Answer- We will give two different approaches. 1) $\overline{AB} = (1/2)\overline{YZ}$, and $\overline{XD} = (1/2)\overline{XC}$. So the area of triangle $ABX = (1/2)\overline{AB} \cdot \overline{XD} = (1/4)[(1/2)\overline{YZ} \cdot \overline{XC}]$=(1/4)area of triangle$YXZ = (1/4) \cdot 8 = 2$. So area of the 4-sided figure $ABZY = 8 - 2$ and since the area of $ADCY$ is one half of the area of $ABZY$, we have the area of $ADCY = (1/2) \cdot 6 = 3$ sq. meters.

2)



Segments $\overline{AF}$ and $\overline{BE}$ are drawn perpendicular to $\overline{YZ}$. Segments $\overline{AB}, \overline{AC}$ and $\overline{BC}$ divide $\triangle XYZ$ in to

4 congruent triangles. Vertical line segments $\overline{AF}, \overline{XC}$ and $\overline{BE}$ divide each of these in half. There are 3 of the eight small triangles in the region $ADCY$, so its area is $(3/8) \cdot 8 = 3$ sq. meters

3. Four new students, Lorna Swain, Benjamin Banneker, Emmy Noether, and Paul Erdös, arrived in Mrs. Brown's homeroom. She looked around the room to consider where to seat them. After a few minutes

she turned to the new students and said, 'There are 1680 ways that you can sit in the empty seats.' How many empty seats are there in Mrs. Brown's room? After she assigns seats to the new students, how many empty seats will remain in her room?

Answer- Let n be the number of empty seats. The number of ways to seat 4 students is $n \cdot n-1 \cdot n-2 \cdot n-3$ and you could set this equal to 1680 and solve for n. But this would be hard. Let us try making a good guess. To start $n \cdot n-1 \cdot n-2 \cdot n-3$ is close to n^4, so take n near $\sqrt[4]{1680}$ or n is close to 6. Then one finds that $8 \cdot 7 \cdot 6 \cdot 5 = 1680$. So there are 8 empty seats in Mrs. Brown's homeroom and after the 4 students are seated there are 4 empty seats.

4. Two ordinary construction ladders of different lengths are placed flush against a vertical wall. In other words, the ladders are vertically upright against the wall. The base of ladder A is then moved 6 feet away from the wall. Doing so lowers the top of the ladder down the side of the wall a distance that is one 1/4-th the length of the ladder itself.

   Repeat the same procedure with the ladder B, except only move the base 5 feet from the wall. Its top point comes down only one-fifth of its length.

   Find the positive difference in the length of the two ladders. Give the value rounded to the nearest whole inch.

   Answer- Let x be the length of the 1st ladder. The equation is $6^2 + (.75x)^2 = x^2$ and solving we find

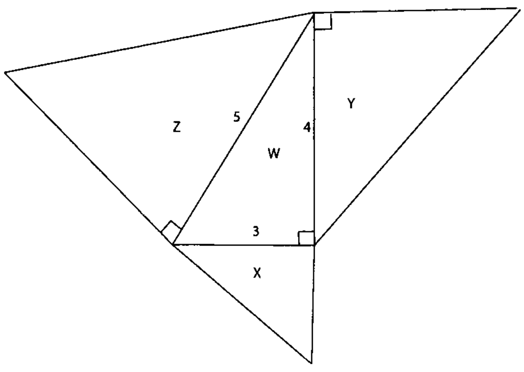
$$x = \frac{27 \cdot \sqrt{7}}{7}.$$

   Let y be the length of the 2nd ladder, the equation is $5^2 + (.8 \cdot y)^2 = y^2$ or $y = 25/3$. So

$$\frac{24 \cdot \sqrt{7}}{7} - \frac{25}{3} = .73781 \cdots$$

   feet or about 8.85 inches. So to the nearest inch the answer is 9 in.

5. Right isosceles triangles are constructed on the sides of a $3-4-5$ right triangle, as shown. A capital letter represents the area of each triangle. Which one of the following is true? Explain why you think so. The picture is not true to length, read the problem carefully and don't let the picture fool you!



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(a) $X + Z = W + Y$

(b) $W + X = Z$

(c) $3X + 4Y = 5Z$

(d) $X + Z = \frac{1}{2}(Y + Z)$

(e) $X + Y = Z$.

The Pythagorean theorem says that if squares are constructed on each side of a right triangle, the sum of the areas of the squares on the legs equals thee area of the square on the hypotenuse. So $2X + 2Y = 2Z$ or $X + Y = Z$. So (E) is right.

Another way to do this problem, is to find $W = \frac{3 \cdot 4}{2} = 6$, $X = \frac{3^2}{2}$, $Y = \frac{4^2}{2} = 8$ and $Z = \frac{5^2}{2}$. Then one sees the only correct answer is (E).