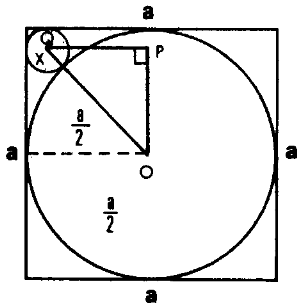
University of Cincinnati Math Competition – 2004
Level II, Answers for Part 2

1.



Let the radius of the required circle be x.
Triangle OPQ is a 45-45 right triangle with legs equal to $\frac{a}{2} - x$ and the hypotenuse is $\frac{a}{2} + x$.

Because the hypotenuse is $\sqrt{2}$ times the leg, we have:

$$\sqrt{2}\left(\tfrac{a}{2} - x\right) = \tfrac{a}{2} + x$$
$$a\sqrt{2} - 2x\sqrt{2} = a + 2x$$
$$2x + 2x\sqrt{2} = a\sqrt{2} - a$$
$$x\left(2 + 2\sqrt{2}\right) = a\left(\sqrt{2} - 1\right)$$
$$x = \frac{\left(\sqrt{2} - 1\right)}{\left(2 + 2\sqrt{2}\right)}a$$

Rationalizing the denominator of the coefficient,

$$x = \frac{\left(\sqrt{2} - 1\right)}{\left(2 + 2\sqrt{2}\right)} \cdot \frac{\left(2 - 2\sqrt{2}\right)}{\left(2 - 2\sqrt{2}\right)}a$$
$$x = \frac{4\sqrt{2} - 6}{4 - 8}a$$
$$x = \frac{3 - 2\sqrt{2}}{2}a$$

2. $f(1) = 0$
$f(2) = 3f(1) + 1 = 1$
$f(3) = 3f(2) + 1 = 3 + 1$
$f(4) = 3f(3) + 1 = 3(3 + 1) + 1 = 3^2 + 3 + 1$
$f(5) = 3f(4) + 1 = 3(32 + 3 + 1) + 1 = 3^3 + 3^2 + 3 + 1$

$f(1003) = 3^{1001} + 3^{1000} + 3^{999} + \ldots + 3 + 1$
$\underline{f((1001) = 3^{999} + 3^{998} + \ldots + 3 + 1}$
$f(1003) - f(1001) = 3^{1001} + 3^{1000}$
$= 3^{1000}(3 + 1)$
$= 4 \cdot 3^{1000}$ or $2^2 \cdot 3^{1000}$

3. $f(2)=\frac{5}{2}$ must be the vertex of the parabola.

$f(x)=a(x-2)^2+\frac{5}{2}$

Substituting the point $\left(5,\frac{17}{2}\right)$,

$$\frac{17}{2}=a(5-2)^2+\frac{5}{2}$$
$$\frac{17}{2}=9a+\frac{5}{2}$$
$$6=9a$$
$$\frac{2}{3}=a$$

So $f(x)=\frac{2}{3}(x-2)^2+\frac{5}{2}$.

To find point(s) of intersection of f(x) and g(x),

$$f(x)=g(x)$$
$$\frac{2}{3}(x-2)^2+\frac{5}{2}=\frac{1}{3}x+b$$
$$\frac{2}{3}\left(x^2-4x+4\right)+\frac{5}{2}-\frac{1}{3}x-b=0$$
$$\frac{2}{3}x^2-\frac{8}{3}x+\frac{8}{3}+\frac{5}{2}-\frac{1}{3}x-b=0$$
$$\frac{2}{3}x^2-3x+\frac{31}{6}-b=0$$

Discriminant must be equal to 0 for point of intersection to be point of tangency.

$$b^2-4ac=0$$
$$(-3)^2-4\left(\frac{2}{3}\right)\left(\frac{31}{6}-b\right)=0$$
$$9-\frac{8}{3}\left(\frac{31}{6}-b\right)=0$$
$$\frac{27}{8}-\frac{31}{6}+b=0$$
$$b=\frac{31}{6}-\frac{27}{8}$$
$$=\frac{124}{24}-\frac{81}{24}$$
$$=\frac{43}{24}$$

4. (a).

$$2\pi(4) - x = 2\pi r$$

$$\frac{8\pi - x}{2\pi} = r$$

$$r^2 + h^2 = 4^2$$

$$h = \sqrt{16 - r^2}$$

$$h = \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$$

$$V(x) = \frac{1}{3}\pi\left(\frac{8\pi - x}{2\pi}\right)^2 \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$$

(b).

$$16 - \left(\frac{8\pi - x}{2\pi}\right)^2 \geq 0$$

$$16 \geq \left(\frac{8\pi - x}{2\pi}\right)^2$$

$$-4 \leq \frac{8\pi - x}{2\pi} \leq 4$$

$$-8\pi \leq 8\pi - x \leq 8\pi$$

$$-16\pi \leq -x \leq 0$$

$$16\pi \geq x \geq 0$$

$$0 \leq x \leq 16\pi$$

But

$r \geq 0$ so $\frac{8\pi - x}{2\pi} \geq 0$

$$8\pi - x \geq 0$$

$$8\pi \geq x$$

$$x \leq 8\pi$$

Domain: $0 \leq x \leq 8\pi$ in keeping with the physical problem

(c). x = 1.25" or 10.30"

5. 1.56% APR $\rightarrow$ 0.13%/month

After 18 months, the value of the money market account is $12{,}000(1.0013)^{18} \approx \$12{,}283.92$.

Overall % gain of 36.5% would result in an overall value of $12{,}000(1.365) = \$16{,}380$.

The money will be in the stock market for 42 months which is 7 compound periods.

Let r = AARR (average annual rate of return) from stock market

$$\left[12{,}000(1.0013)^{18}\right]\left(1+\tfrac{r}{2}\right)^7 = 12{,}000(1.365)$$

$$\left(1+\tfrac{r}{2}\right)^7 = \frac{1.365}{1.0013^{18}}$$

$$1+\tfrac{r}{2} = \left(\frac{1.365}{1.0013^{18}}\right)^{\frac{1}{7}}$$

$$\tfrac{r}{2} = \left(\frac{1.365}{1.0013^{18}}\right)^{\frac{1}{7}} - 1$$

$$r = 2\left[\left(\frac{1.365}{1.0013^{18}}\right)^{\frac{1}{7}} - 1\right]$$

$$r \approx .083933$$

$$\therefore r \approx 8.39\%$$