Morita Equivalence of Primitive Factors of $U(\mathbf{sl}(2))$

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Abstract. The primitive factors of the enveloping algebra $U(\mathbf{sl}(2,\mathbf{C})$ are classified up to Morita equivalence using the Hattori-Stallings map.

Let $U = U(\mathbf{sl}(2,\mathbf{C}))$ be the enveloping algebra of the Lie algebra $\mathbf{sl}(2,\mathbf{C})$. The minimal primitive ideals of U are of the form $(\Omega - \alpha)$ where $\alpha \in \mathbf{C}$ and Ω is the Casimir element. In [3] Dixmier proved that the algebras $B_\alpha = U/(\Omega - \alpha)$ are non-isomorphic as $\mathbf{C}$-algebras. On the other hand, Stafford showed in [9] that many of the B_α are Morita equivalent via "translation functors" and asked whether it was true that all the simple B_α are equivalent. In this note we answer this question by showing that two such algebras are equivalent only if there is a translation functor defining equivalence. In passing we give a new and very short proof of Dixmier's result. The essential idea is to look at the Hattori-Stallings traces of the rank one projective modules.

The parameterization given above for the B_α is an unnatural one to work with in this context. By Duflo's theorem each minimal primitive is the annihilator of a Verma module $M(\lambda)$ for $\lambda \in \mathbf{C}$ (here we are making the natural identification of $\mathbf{h}^*$ with $\mathbf{C}$ via $\nu \longmapsto \nu(H)$). For this and other basic information, we refer the reader to [4]. Define D_λ to be $U/\mathrm{Ann}_U M(\lambda)$. Recall that $\mathrm{Ann}_U M(\lambda) = (\Omega - (\lambda^2 - 1))$ and hence that $\mathrm{Ann}_U M(\lambda) = \mathrm{Ann}_U M(\mu)$ if and only if $\lambda = \pm\mu$. Thus Dixmier's theorem states in this context that $D_\lambda \simeq D_\mu$ if and only if $\mu = \pm\lambda$. On the other hand, Stafford showed in [9] that D_μ is Morita equivalent to $D_{\mu+1}$ whenever $\mu \neq 0, -1$. Thus for each fixed $\lambda \notin \mathbf{Z}$, the algebras $\{D_{n\pm\lambda} : n \in \mathbf{Z}\}$ are Morita equivalent. We show that these, together with the sets $\{D_n : 0 \neq n \in \mathbf{Z}\}$ and $\{D_0\}$ are the $\mathbf{C}$-linear Morita equivalence classes. (Recall that a $\mathbf{C}$-linear equivalence is one given by a $\mathbf{C}$-functor; that is, a functor F such that the maps from $\mathrm{Hom}(A,B)$ to $\mathrm{Hom}(FA,FB)$ are $\mathbf{C}$-linear maps.) Using global dimension we may dispose of the integer cases immediately. For it is shown in [9] that

$$\mathrm{gldim}\, D_\lambda = \begin{cases} \infty & \text{if } \lambda = 0 \\ 2 & \text{if } \lambda \in \mathbf{Z}\backslash\{0\} \\ 1 & \text{if } \lambda \in \mathbf{C}\backslash\mathbf{Z} \end{cases}$$

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Thus D_0 can be equivalent to no other D_λ. Similarly the D_n for $n \in \mathbf{Z}\backslash\{0\}$ are equivalent to each other but to no other D_λ.

Let us briefly review the definition of the Hattori-Stallings map that we will be using for the calculations below. Let R be a ring, let $K_0(R)$ be the Grothendieck group and let $H_0(R) = R/[R,R]$ be the trace group. Let P be a projective left R-module. Then there exist $f_1, ..., f_n \in P^* = \mathrm{Hom}_R(P,R)$ and $x_1, ..., x_n \in P$ such that $1 = \sum_{i=1}^n f_i \otimes x_i \in P^* \otimes_R P \simeq \mathrm{End}_R P$. The trace of P is defined to be the element $tr(P) = \sum_{i=1}^n f_i(x_i) + [R,R]$ of $H_0(R)$ and this element is independent of the choice of the f_i and x_i. Since tr is additive on direct sums it induces a map $Tr : K_0(R) \to H_0(R)$ given by $Tr([P]) = tr(P)$. See [1,5,10] for further details.

It is well-known and fairly straightforward to check that $D_\lambda = \mathbf{C}\cdot 1 \oplus [D_\lambda, D_\lambda]$ and hence the natural map $\pi : \mathbf{C} \to H_0(D_\lambda)$ is an isomorphism [3;1.10]. It is also known that $K_0(D_\lambda) \simeq \mathbf{Z} \oplus \mathbf{Z}$. Since most of the proofs of this result in the literature involve the theory of D-modules, we give here an outline of an elementary proof of this fact.

First recall the standard realization of D_λ as a subring of the Weyl algebra $A_1(\mathbf{C})$. Let p and q be generators of $A_1(\mathbf{C})$ such that $[p,q] = 1$. Define:

$$e = -p, \quad f = q(qp - \lambda + 1), \quad h = -2qp + \lambda - 1.$$

Then there is a well-defined map from $U(\mathbf{sl}(2,\mathbf{C}))$ to $A_1(\mathbf{C})$ sending the canonical generators E, F and H to e, f and h respectively. The kernel of this map is $(\Omega - (\lambda^2 - 1)) = \mathrm{Ann}M(\lambda)$. Thus D_λ may be identified with the subalgebra of $A_1(\mathbf{C})$ generated by e, f and h. The following proof was also outlined in [6; 2.5] where more details may be found concerning the K-theoretical ideas involved.

Proposition 1. *If $\lambda \neq 0$, then $K_0(D_\lambda) \simeq \mathbf{Z} \oplus \mathbf{Z}$. Furthermore if $\lambda \notin \{0, -1, -2, ...\}$ then the classes of D_λ and the projective ideal $P_\lambda = D_\lambda p + D_\lambda(qp - \lambda)$ form a basis for $K_0(D_\lambda)$.*

Proof. Assume that $\lambda \notin \{0, -1, -2, ...\}$. Then it can be shown that $A_1(\mathbf{C})$ is a flat epimorphic extension of D_λ; that is, it lies between D_λ and its quotient division ring and is flat as a right D_λ-module. Furthermore, the class of torsion modules is generated by $M(-\lambda) = D_\lambda/P_\lambda$. It then follows from Quillen's localization sequence [6] that the Grothendieck group $G_0(D_\lambda)$ of the category of finitely generated left D_λ-modules is a free abelian group with basis the classes $[D_\lambda]$ and $[M(-\lambda)]$. Because D_λ has finite global dimension in this situation, the Cartan map is an isomorphism and the result follows.

Let $Q(D_\lambda)$ be the quotient division algebra of D_λ, let $rk : K_0(D_\lambda) \to \mathbf{Z}$ be the usual rank function given by $rk([P]) = \mathrm{length}(Q(D_\lambda) \otimes_{D_\lambda} P)$ and define $Rk_0(D_\lambda) = \mathrm{Ker}(rk)$. From Proposition 1 it is clear that $Rk_0(D_\lambda) \simeq \mathbf{Z}$.

Lemma 2. *Let ξ be a generator for $Rk_0(D_\lambda)$. Then $Tr(\xi) = \pm\lambda^{-1}$.*

Proof. We know from the proposition that the generators of $Rk_0(D_\lambda)$ are $\pm([P_\lambda] - [D_\lambda])$. Recall that $P_\lambda = D_\lambda p + D_\lambda(qp - \lambda)$ and identify P_λ^* as $\{x \in Q(D_\lambda) \mid P_\lambda x \subset D_\lambda\}$. Since $qp + (-1)(qp - \lambda) = \lambda$, we have that in

$P_\lambda^* \otimes P_\lambda$,

$$1 = \lambda^{-1}(q \otimes p + (-1) \otimes (qp - \lambda))$$

Thus $Tr([P_\lambda]) = \lambda^{-1}(pq - (qp - \lambda)) = (\lambda + 1)\lambda^{-1}$. Hence, since $Tr([D_\lambda]) = 1$, it follows that $Tr(\xi) = \pm\lambda^{-1}$, as required.

THEOREM 3 [3]. *Let $\lambda, \mu \in \mathbf{C}$. Then D_λ is isomorphic to D_μ as a $\mathbf{C}$-algebra if and only if $\mu = \pm\lambda$.*

PROOF. Let $\phi : D_\lambda \to D_\mu$ be a $\mathbf{C}$-algebra isomorphism. Then ϕ induces maps $K_0(\phi)$ and $H_0(\phi)$ such that the following diagram commutes [1]:

$$\begin{array}{ccc} K_0(D_\lambda) & \xrightarrow{Tr} & H_0(D_\lambda) \\ \downarrow K_0(\phi) & & \downarrow H_0(\phi) \\ K_0(D_\mu) & \xrightarrow{Tr} & H_0(D_\mu) \end{array}$$

Furthermore, after we have identified $H_0(D_\lambda)$ and $H_0(D_\mu)$ with $\mathbf{C}$ in the manner described above, the induced map $H_0(\phi) : H_0(D_\lambda) \to H_0(D_\mu)$ is just the identity map. Let ξ be a generator of $Rk_0(D_\lambda)$ such that $Tr(\xi) = \lambda^{-1}$. Since rank commutes with $K_0(\phi)$, we must have that $K_0(\phi)(\xi)$ is a generator of $Rk_0(D_\lambda)$. But then

$$\lambda^{-1} = Tr(\xi) = Tr(K_0(\phi)(\xi)) = \pm\mu^{-1}.$$

Hence $\mu = \pm\lambda$, as required.

We now turn to the problem of Morita equivalence. Given a Morita equivalence between D_λ and D_μ, we get a commutative diagram similar to that used above in Theorem 3. The difference in this case is that the induced map on the H_0's is no longer the identity.

LEMMA 4. *Suppose that $F : D_\lambda$-Mod $\to D_\mu$-Mod is a $\mathbf{C}$-linear equivalence of categories. Then $H_0(F) : H_0(D_\lambda) \to H_0(D_\mu)$ is given by multiplication by $(\mu + n)\mu^{-1}$ for some integer n.*

PROOF. By standard Morita theory, F is given by $P \otimes_{D_\lambda}$ - for some projective generator P in Mod-D_λ with $\mathrm{End}_{D_\lambda}(P) = D_\mu$. Since D_λ and D_μ are both Noetherian domains, the rank of P must be one as both a right D_λ-module and as a left D_μ-module. Since F is $\mathbf{C}$-linear, $H_0(F)$ is a $\mathbf{C}$-linear map; hence it suffices to calculate $H_0(F)(1)$. Now the map $H_0(F)$ is given by sending the coset of the element $\sum f_i \otimes a_i \in P^* \otimes_{D_\mu} P \simeq D_\lambda$ to the coset of the element $\sum a_i \otimes f_i \in P \otimes_{D_\lambda} P^* \simeq D_\mu$ [2; p.48]. Hence, in particular, $H_0(F)(1) = Tr([{}_{D_\mu}P])$. Since ${}_{D_\mu}P$ is a rank one projective D_μ-module, it follows from Lemma 2 that $Tr([{}_{D_\mu}P] - [D_\mu]) = n\mu^{-1}$ for some $n \in \mathbf{Z}$. Hence $Tr([{}_{D_\mu}P]) = 1 + n\mu^{-1} = (\mu + n)\mu^{-1}$. Thus $H_0(F)$ is given by multiplication by $(\mu + n)\mu^{-1}$.

THEOREM 5. *Let λ and μ be elements of $\mathbf{C}\backslash\{0\}$. Then D_λ and D_μ are $\mathbf{C}$-linearly Morita equivalent if and only if $\mu = m \pm \lambda$ for some $m \in \mathbf{Z}$.*

PROOF. ($\Leftarrow$) This follows from [9; Cor. 3.3].

($\Rightarrow$) Let $F : D_\lambda$-Mod $\to D_\mu$-Mod be a $\mathbf{C}$-linear Morita equivalence. It is well-known (see for example [8]) that F induces the following commutative diagram (where the vertical maps are isomorphisms):

$$\begin{array}{ccc} K_0(D_\lambda) & \xrightarrow{Tr} & H_0(D_\lambda) \\ \downarrow {\scriptstyle K_0(F)} & & \downarrow {\scriptstyle H_0(F)} \\ K_0(D_\mu) & \xrightarrow{Tr} & H_0(D_\mu) \end{array}$$

Now $K_0(F)$ must preserve rank. Hence it maps $Rk_0(D_\lambda)$ isomorphically onto $Rk_0(D_\mu)$. Let $\eta = K_0(F)(\xi)$ be a generator of $Rk_0(D_\mu)$ with $Tr(\eta) = \mu^{-1}$. Then using the commutativity of the above diagram we obtain that

$$\mu^{-1} = Tr(\eta) = H_0(F)(Tr(\xi)) = \pm\lambda^{-1}(\mu + n)\mu^{-1}$$

for some $n \in \mathbf{Z}$. Hence $\mu + n = \pm\lambda$.

We have tried in this article to take a non-technical approach, avoiding in particular the theory of D-modules. However all of the ideas involved are special cases of general concepts applicable to the case of an arbitrary semisimple Lie algebra. One may make similar conjectures about isomorphism and Morita equivalence classes of primitive factors in this general situation. Preliminary calculations suggest that although this technique of looking at traces can go a long way towards distinguishing between these factors, the question cannot be completely solved using this approach.

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